MIDTERM

Instructions

Open book, open notes. You may appeal to any result stated in Do Carmo. References to theorems from other sources will not be accepted.

There are **five** questions. The time limit is **four hours**. No credit will be given for work done after four hours.

Hand in the midterm to the usual box before 4pm on Wednesday, February 10^{th}.

Turn over when beginning exam.

1. Let $\alpha : I \to \mathbb{R}^3$ be a regular curve, not necessarily parametrized by arc length. Let $\beta : J \to \mathbb{R}^3$ be a reparametrization of $\alpha(I)$ by arc length measured from $t_0 \in I$ (see do Carmo, pp. 21–22). Let $t = t(s)$ be the inverse function of s and set $\frac{d\alpha}{dt} = \alpha'$, $\frac{d^2\alpha}{dt^2} = \alpha''$, etc. Prove that:

   (a) $\frac{dt}{ds} = \frac{1}{||\alpha'||}$, and $\frac{d^2t}{ds^2} = -\left(\frac{\alpha' \cdot \alpha''}{||\alpha'||^4}\right)$;

   (b) the curvature of α at $t \in I$ is

   $$k(t) = \frac{||\alpha' \wedge \alpha''||}{||\alpha'||^3} \ ;$$

   (c) the torsion of α at $t \in I$ is

   $$\tau(t) = -\frac{(\alpha' \wedge \alpha'') \cdot \alpha'''}{||\alpha' \wedge \alpha''||^2} \ .$$

   Hint. Compute $(\beta' \wedge \beta'') \cdot \beta'''$.

2. Let $f(x, y, z) = (x + y + z - 1)^2$.

   (a) Locate the critical points and critical values of f.

   (b) For what values of c is the set $f(x, y, z) = c$ a regular surface?

   (c) Answer the questions of parts (a) and (b) for the function $f(x, y, z) = xyz^2$.

3. Let $A \subseteq S$ be a subset of a regular surface S. Prove that A is itself a regular surface if and only if A is open in S; that is, $A = U \cap S$ where U is an open set in $\mathbb{R}^3$.

4. Suppose that S is a regular surface in $\mathbb{R}^3$ and let P be a plane in $\mathbb{R}^3$. Suppose that all points of S lie on the same side of P and that $P \cap S \neq \varnothing$. Prove that P is tangent to S at all points of $P \cap S$.

5. Let S be a regular surface in $\mathbb{R}^3$ and let $p_0 \in S$. Define $f : S \to \mathbb{R}$ by $f(p) = ||p - p_0||^2$. Prove that

   $$df_p(w) = 2w \cdot (p - p_0)$$

   for $w \in T_p(S)$.