Quantum trajectories and the appearance of particle tracks in detectors

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Vignette – Measurement of position

**Single measurement:** For a state $\psi \in L^2(\mathbb{R}^d)$ what is the probability of finding the particle in a box $E$?

$$P(x \in E) = \int_E |\psi(x)|^2 \, dx.$$ 

**Repeated Measurement:** Consider a protocol:

1. Measure the position (record $x$)
2. Evolve freely by $H = P^2$ for time $t$
3. Measure the position again (record $x_t$)

What is the probability $P(x \in E, x_t \in F)$?

There is no (unique) answer to the question. **We need a formalism that conveniently encodes all the answers.**
Kraus Operator Formalism

Objects:

- Set $\Omega$ of possible measurement results $q$.
- Prior measure $\mu$ (and sigma algebra $\mathcal{F}$) on $\Omega$.
- Hilbert space $\mathcal{H}$ of the system.

Definition (Kraus operators $V_q$)

A measurable function $q \in \Omega \to V_q \in B(\mathcal{H})$ satisfying

$$\int_{\Omega} V_q^* V_q d\mu(q) = 1.$$ 

Meaning: Given a measurement result $q$ the state jumps as $\psi \to V_q \psi$:

1. $\mathbb{P}(q \in E) = \int_E \|V_q \psi\|^2 = \int_E (\psi, V_q^* V_q \psi)$,
2. The normalized state after the measurement is $\frac{V_q \psi}{\|V_q \psi\|}$. 
Example (Projection Measurements)
\( \Omega = \{1, \ldots, k\}, \ V_j = P_j, \) projections \( P_j \) form an orthogonal decomposition of identity, \( P_1 + \cdots + P_k = 1. \)

Example (Unitary Evolution)
\( \Omega = \{1\} \) and \( V_1 \) is unitary.
Example (A measurement of position)
\[ \Omega = \mathbb{R}, \quad V_q = \frac{1}{(\sigma \sqrt{2\pi})^d} \exp\left(-\frac{1}{2} \frac{(X-q)^2}{2\sigma^2}\right), \quad X \text{ is the position operator.} \]
For the average measurement result and the variance we get:

\[ \int_{\mathbb{R}^d} q \| V_q \psi \|^2 \, dx = (\psi, X\psi), \quad \int_{\mathbb{R}} q^2 \| V_q \psi \|^2 \, dx = (\psi, X^2\psi) + \sigma^2. \]

Example (General Measurement of Position)
\[ \Omega = \mathbb{R}, \quad V_q = \sqrt{p(X - q)} \text{ where } p \text{ is a probability distribution.} \]

What is the probability \( \mathbb{P}(q \in E, q_t \in F) \)?

\[ \mathbb{P}(q \in E, q_t \in F) = \int_{q \in E, q_t \in F} \| V_{q_t} e^{-itP^2} V_q \psi \|^2 \, dq \, dq_t. \]
Quantum Trajectories

Given Kraus operators $V_q$, the map

$$(q_1, \ldots, q_n) \rightarrow V_{q_n} \cdots V_{q_1} \psi$$

is the quantum trajectory.

Basic Objects:

1. $\mathbb{P}_\psi(q_1 \in E_1, \ldots, q_n \in E_n) = \int_{E_1 \times \cdots \times E_n} \|V_{q_n} \cdots V_{q_1} \psi\|^2$ defines a probability measure on $\Omega^N$.

2. Process $\psi_n := \frac{V_{q_n} \cdots V_{q_1} \psi}{\|V_{q_n} \cdots V_{q_1} \psi\|}$ is a Markov process on $(\Omega^N, \mathbb{P}_\psi)$.

Quantum trajectories theory studies limit properties of $\mathbb{P}_\psi$ and $\psi_n$. 
Setting up the theory: Kraus, Davies, ... 60-70’s
Quantum Stochastic Calculus & Filtering theory: Hudson, Parthasarathy & Belavkin, Holevo 80-90’s
Kümmerer Maassen: Purification of quantum trajectories [2006]
Non-demolition case: Bauer, Bernard [2011], ..., Ballesteros, Crawford, F, Fröhlich, Schubnel [2018]
Uniqness of invariant measure: Benoist, F, Pautrat, Pellefrini [2019]
Entropy Production: Benoist, Jakšič, Pautrat, Pillet [2018-2020]

In Quantum Optics: Cohen-Tannoudji, ..., Haroche.
For \( f \in L^2(\mathbb{R}^2) \) let
\[ X_j f(x) := x_j f(x), \quad P_j f(x) = i \partial_j f(x), \]
and for \( q \in \Omega = \mathbb{R}^2 \) put
\[ V_q = \frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{(X - q)^2}{4\sigma^2}\right) \exp\left(-it \frac{P^2}{2M}\right) \]
Example

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Sampling $q = q_1, q_2, \ldots$:
Example

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The same from a cloud chamber:
Debate @ Solvay [1927] Einstein’s question: “A radioactive sample emits $\alpha$-particles in all directions; these are made visible by the method of the Wilson cloud chamber. Now, if one associates a spherical wave with each emission process, how can one understand that the track of each $\alpha$-particle appears as a (very nearly) straight line? In other words: how can the corpuscular character of the phenomenon be reconciled here with the representation by waves?”

Born, Heisenberg, Gamow answer: “every determination of position [with uncertainty $\lambda$] reduces therefore the wave packet back to its original size $\lambda$”

Mott [1929] Taking into account interactions with the gas atoms, the probability that two atoms are ionized by passage of the particle is substantially different from zero only if the atoms are in a cone of small angle emanating from the origin. (see Figari-Teta [2014]).
Setup

We consider a density matrix $\rho$ on

$$\mathcal{H} = L^2(\mathbb{R}^d, d^d x).$$

The position- and momentum operators of the particle are given by

$$X := (X_1, \ldots, X_d) \quad \text{and} \quad P := (P_1, \ldots, P_d).$$

Measurement of position is modelled by

$$V_q \equiv V_q(X) := \frac{1}{\left[(2\pi)^d \det \Sigma\right]^{\frac{1}{4}}} \exp \left\{ \frac{-1}{4} (X - q) \Sigma^{-1} (X - q)^t \right\},$$

where $\Sigma$ is a positive-definite $d \times d$. 
We assume that the particle evolves quasi-freely with a symplectic matrix

\[ S = \begin{pmatrix} S_{xx} & S_{xp} \\ S_{px} & S_{pp} \end{pmatrix}, \quad \text{on} \quad \Gamma := \mathbb{R}^d \oplus \mathbb{R}^d, \]

\( S \) determines a unitary operator, \( U_S \), on \( \mathcal{H} \) with the property that

\[ U_S^* (X, P) U_S = (X, P) \cdot S^t. \]

The evolution is

\[ W_n(q_n) := U_S V_{q_n}(X) \ldots U_S V_{q_0}(X) \]

A probability measure, \( d\mathbb{P}_\rho^{(n)} \), on \( (\mathbb{R}^d)^{n+1} \) is introduced by setting

\[ d\mathbb{P}_\rho^{(n)}(q_0, \ldots, q_n) = \rho(W_n(q_n)^* W_n(q_n)) dq_n \]
Assumptions

We require that the $d \times d$ matrix $S_{xp}$ is invertible.

**Lemma:** The equation

$$W \cdot S_{xx} - S_{px} = (S_{pp} - W \cdot S_{xp}) \left( W - \frac{i}{2} \Sigma^{-1} \right),$$

for an unknown $d \times d$ matrix $W$ has a solution, $W = \hat{W}$, with the following properties:

$$\hat{W} = \hat{W}^t, \quad \text{Im} \hat{W} > 0, \quad \text{and} \quad \left(2\text{Im} \hat{W}\right)^{-1} < \Sigma.$$

**Lemma:** Manifold of coherent states $|\hat{W}, \zeta\rangle$, $\zeta \in \Gamma$ with squeezing parameter $\hat{W}$ is preserved by the time evolution.
Stochastic Process on the Phase Space $\Gamma$

Put $K : \mathbb{R}_x^d \rightarrow \Gamma = \mathbb{R}_x^d \oplus \mathbb{R}_p^d,$

$$K = \left( \frac{1}{\text{Re} \hat{W}} \right) (2\text{Im} \hat{W})^{-1} (\Sigma - (2\text{Im} \hat{W})^{-1})^{-1}.$$ 

Stochastic process $(\zeta_n)_{n \in \mathbb{N}_0}$, $\zeta_n \in \Gamma$, $\forall n \in \mathbb{N}_0$: 

$$\zeta_{n+1} = S(\zeta_n - K\eta_n), \quad n = 0, 1, 2, \ldots ,$$

where $(\eta_n)_{n \in \mathbb{N}_0}$ are i.i.d. Gaussian random vectors in $\mathbb{R}_x^d$ with mean 0 and covariance given by the matrix $\Sigma - (2\text{Im} \hat{W})^{-1}$. The law of the initial random variable $\zeta = \zeta_0$ is given by the probability measure

$$\rho(|\hat{W}, \zeta\rangle\langle \hat{W}, \zeta|)d\lambda(\zeta) = \langle \hat{W}, \zeta | \rho | \hat{W}, \zeta \rangle \frac{d^{2d} \zeta}{(2\pi)^d}.$$
Denote $\xi := [\zeta]$, with $\zeta = (\xi, \pi)$.

**Theorem**

Let $d_{\mathbb{P}_\rho}$ be the law of the sequence, $Q_\infty$, of random variables whose values, $q_\infty$. Let $(\zeta_n)_{n \in \mathbb{N}_0}$ be the stochastic process introduced above. Then the following “equality in law” holds:

$$Q_\infty = \left(\xi_n + \eta_n\right)_{n \in \mathbb{N}_0}, \quad \text{where} \quad \xi_n = [\zeta_n].$$

If the density matrix $\rho$ is such that the expectation values $\rho(|X|)$ and $\rho(|P|)$ exist then

$$\mathbb{E}_\rho(Q_n) = \left[(\rho(X), \rho(P))(S^t)^n\right],$$

where

$$\mathbb{E}_\rho(F) := \int d_{\mathbb{P}_\rho}(q_\infty)F(q_\infty).$$
Thank you!