Weyl formulae for Schrödinger operators with critically singular potentials

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Joint work with Christopher D. Sogge and Cheng Zhang
Consider the Schrödinger operators

\[ H_V = -\Delta g + V(x) \]

on smooth compact \( n \)-dimensional Riemannian manifolds \((M, g)\), where \(-\Delta g\) denotes the Laplace-Beltrami Operator. We shall assume throughout that the potentials \( V \) are real-valued. Moreover, we shall assume that

\[ V \in L^1(M), \quad \text{and} \quad V^- = \max\{0, -V\} \in \mathcal{K}(M), \]

where \( \mathcal{K}(M) \) denotes the Kato class.
Some background

Recall that $\mathcal{K}(M)$ is all $V$ satisfying

$$\lim_{\delta \to 0} \left( \sup_{x \in M} \int_{B(x, \delta)} |V(y)| \, h_n(d_g(x, y)) \, dy \right) = 0,$$

where $d_g$, $dy$ and $B(x, \delta)$ denote geodesic distance, the volume element and the geodesic ball of radius $\delta$ about $x$ associated with the metric $g$ on $M$, respectively, and

$$h_n(r) = \begin{cases} r^{2-n}, & n \geq 3 \\ \log(2 + 1/r), & n = 2. \end{cases}$$

Note that $\mathcal{K}(M) \subset L^1(M)$ while $L^p(M) \subset \mathcal{K}(M)$ for $p > \frac{n}{2}$. 
Some background

if $V$ satisfy the condition above then the Schrödinger operators $H_V$ is self-adjoint and bounded from below. Additionally, in this case, since $M$ is compact, the spectrum of $H_V$ is discrete. Also, the associated eigenfunctions are continuous. After possibly adding a constant $V$, we shall assume that $H_V$ is a positive operator, and write the spectrum of $\sqrt{H_V}$ as

$$\{\tau_k\}_{k=1}^{\infty},$$

where the eigenvalues, $\tau_1 \leq \tau_2 \leq \cdots$, are arranged in increasing order and we account for multiplicity. For each $\tau_k$ there is an eigenfunction $e_{\tau_k} \in \text{Dom } (H_V)$ (the domain of $H_V$) so that

$$H_V e_{\tau_k} = \tau_k^2 e_{\tau_k}.$$

We shall always assume that the eigenfunctions are $L^2$-normalized, i.e.,

$$\int_M |e_{\tau_k}(x)|^2 \, dx = 1.$$
By the heat kernel estimates of Li and Yau \((V \in C^\infty)\) and Sturm \((V \in \mathcal{K}(M))\), for \(0 < t \leq 1\) there is a uniform constant \(c = c_{M,V} > 0\) so that

\[
(e^{-tH_V})(x,y) \lesssim \begin{cases} 
  t^{-n/2} \exp(-c(d_g(x,y))^2/t), & \text{if } d_g(x,y) \leq \text{Inj} (M)/2, \\
  1 & \text{otherwise.}
\end{cases}
\]

- We shall only require the \(V^- \in \mathcal{K}(M)\) since by the Feynman–Kac formula the \(V^+\) will not make the kernel bound larger.
If
\[ H^0 = -\Delta_g + 1 \]
denote the unperturbed operator with
\[ H^0 e_j^0 = \lambda_j^2 e_j^0, \quad \text{and} \quad \int_M |e_j^0(x)|^2 \, dx = 1, \]
then one has the “sharp Weyl formula”
\[ N^0(\lambda) = (2\pi)^{-n} \omega_n \text{Vol}_g(M) \lambda^n + O(\lambda^{n-1}), \quad N^0(\lambda) = \#\{j : \lambda_j \leq \lambda\}, \]
where \( \omega_n \) denotes the volume of the unit ball in \( \mathbb{R}_n \) and \( \text{Vol}_g(M) \) denotes the Riemannian volume of \( M \).

**History**
- Weyl (1911)
- Levitan (1952)
- Avakumović (1952 & 1956)
- Hörmander (1968)
Some background

Pointwise Sharp Weyl Formula

\[ \mathbb{1}(-\Delta_g + 1 \leq \lambda^2)(x, x) = (2\pi)^{-n} \omega_n \lambda^n + O(\lambda^{n-1}), \text{ uniformly in } x \in M. \]

The above bounds cannot be improved for the standard round sphere, which accounts for the nomenclature “sharp Weyl formula”.
An abstract universal bound

Proposition 1.

Let $V \in L^1(M)$, and $V^- \in \mathcal{K}(M)$, let $H_V$ as above, if $N^0(\lambda)$ satisfies

$$N^0(\lambda) = \int_M \sum_{\lambda_j \leq \lambda} |e_j^0(x)|^2 \, dx = (2\pi)^{-n} \omega_n \text{Vol}_g(M) \lambda^n + O(\varepsilon \lambda^{n-1}),$$

where $\varepsilon = \varepsilon(\lambda)$ is a non-increasing function in $\lambda$ which satisfies $\varepsilon(2\lambda) \geq \frac{1}{2}\varepsilon(\lambda)$ and $0 < \varepsilon(\lambda) \leq 1$, $\forall \lambda \geq 1$. Then we have

$$N_V(\lambda) = (2\pi)^{-n} \omega_n \text{Vol}_g(M) \lambda^n + O(\varepsilon \lambda^{n-1} + \varepsilon^{-2} \lambda^{n-\frac{3}{2}}).$$
Main results

**Theorem 2.**

Let $V \in L^1(M)$, and $V^- \in \mathcal{K}(M)$, let $H_V$ as above and set

$$N_V(\lambda) = \# \{ k : \tau_k \leq \lambda \}.$$

We then have

$$N_V(\lambda) = (2\pi)^{-n} \omega_n \text{Vol}_g(M) \lambda^n + O(\lambda^{n-1}).$$

- For smooth potentials $V$, this just follows from Hörmander(1968)
Main results, cont’d

Theorem 3.

Let $V \in L^1(M)$, and $V^- \in K(M)$, let $H_V$ be as above, and assume that the set $C$ of directions of periodic geodesics has measure zero in $S^* M$. Then

$$N_V(\lambda) = (2\pi)^{-n} \omega_n \text{Vol}_g(M) \lambda^n + o(\lambda^{n-1}).$$

- When $V = 1$ or $V$ smooth, this is due to Duistermaat Guillemin (1975)
- When $M$ is a product manifold, Iosevich Wyman (2019) showed that $M$ satisfies the condition above, Canzani Galkowski (2020) further improved the $o(\lambda^{n-1})$ error term for product manifolds.
Main results, cont’d

**Theorem 4.**

Assume that the sectional curvatures of \((M, g)\) are non-positive. Then, if \(V \in L^1(M)\), and \(V^- \in K(M)\),

\[
N_V(\lambda) = (2\pi)^{-n} \omega_n \text{Vol}_g(M) \lambda^n + O(\lambda^{n-1}/\log \lambda).
\]

- When \(V = 1\) or \(V\) smooth, this is due to Bérard(1977)
The flat torus

Also, when $V \equiv 1$, the Weyl counting function then just counts the number of integer lattice points lying in the ball of radius $\lambda$

$$N^0(\lambda) = (2\pi)^{-n} \omega_n \text{Vol}_g(M) \lambda^n + r_n(\lambda),$$

where

$$r_n(\lambda) \lesssim \begin{cases} 
\lambda^{n-2}, & \text{if } n \geq 5 \\
\lambda^2 (\log \lambda)^{2/3}, & \text{if } n = 4 \\
\lambda^{21/16} + \epsilon, & \text{if } n = 3 \\
\lambda^{131/208} (\log \lambda)^{18627/6320}, & \text{if } n = 2.
\end{cases}$$

Main results, cont’d

**Theorem 5.**

Let $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ denote the standard torus with the flat metric, and assume that $V \in \mathcal{K}(M)$ when $n = 2$ and $V \in L^p(M)$, $V^- \in \mathcal{K}(M)$ for some $p > \frac{2n}{n+2}$ if $n \geq 3$. Then

$$N_V(\lambda) = (2\pi)^{-n}\omega_n\text{Vol}_g(M)\lambda^n + O(\lambda^{n-2+2/(n+1)}). \quad (1)$$

Moreover, if $V \in L^2(M)$ and $V^- \in \mathcal{K}(M)$, we have for $n \geq 4$,

$$N_V(\lambda) = (2\pi)^{-n}\omega_n\text{Vol}_g(M)\lambda^n + O(\lambda^{n-2+\varepsilon}). \quad (2)$$

- When $V \equiv 1$, the above bounds (1) are the classical results of Hlwaka(1950).
Main results

Theorem 6 (Pointwise Weyl law).

Let \( n \geq 2 \) and \( V \in \mathcal{K}(M) \). Then for any fixed \( \epsilon > 0 \) there exists a \( \Lambda(\epsilon, V) < \infty \) such that for \( \lambda > \Lambda(\epsilon, V) \), we have

\[
\sup_{x \in M} \left| \mathbb{1}_\lambda(P_V)(x, x) - \frac{\omega_n}{(2\pi)^n} \lambda^n \right| \leq C_V \epsilon \lambda^{n}.
\]

So as \( \lambda \to \infty \) and uniformly in \( x \in M \),

\[
\mathbb{1}_\lambda(P_V)(x, x) = \frac{\omega_n}{(2\pi)^n} \lambda^n + o(\lambda^n).
\]

- When \( n = 3 \), Frank-Sabin (2020) proved this under the same condition, and they showed for any \( \eta > 0 \), there exists \( V \in \mathcal{K}(M) \) such that the error term cannot be replaced by \( O(\lambda^{3-\eta}) \).
- Thus, the error term \( o(\lambda^{n}) \) is expected to be “sharp”.
Main results, cont’d

**Theorem 7 (Sharp pointwise Weyl law).**

Let $n \geq 2$ and $V \in L^n(M)$. Then for $\lambda \geq 1$

$$
\sup_{x \in M} |\mathbb{1}_\lambda(P_V)(x, x) - \frac{\omega_n}{(2\pi)^n} \lambda^n| \leq C_V \lambda^{n-1}.
$$

So uniformly in $x \in M$,

$$
\mathbb{1}_\lambda(P_V)(x, x) = \frac{\omega_n}{(2\pi)^n} \lambda^n + O(\lambda^{n-1}).
$$

- For smooth potentials $V$, this just follows from Hörmander (1968)
- Frank-Sabin (2020) proved the same estimates under a different condition: $V$ satisfies for some $\epsilon' > 0$

$$
\sup_{x \in M} \int_{d_g(y,x) < \epsilon'} \frac{|V(y)|}{d_g(y,x)^2} dy < \infty, \quad (3)
$$

The condition (3) is satisfied by $V \in L^q(M)$, for any $q > 3$. 
Main approach

We shall follow the classical approach of rewriting the trace of

\[ E_\lambda(x, y) = \sum_{\tau_k \leq \lambda} e_{\tau_k}(x)e_{\tau_k}(y), \]

by using the wave equation. Since the Fourier transform of the indicator function \( \mathbb{1}_\lambda(\tau) \) is \( 2\frac{\sin \lambda t}{t} \), we have

\[ N^V(\lambda) = \int_M \sum_{\tau_k \leq \lambda} |e_{\tau_k}(x)|^2 \, dx \]

\[ = \frac{1}{\pi} \int_M \int_{-\infty}^{\infty} \frac{\sin t\lambda}{t} (\cos(tP_V))(x, x) \, dtdx, \]

if

\[ (\cos(tP_V))(x, y) = \sum_k \cos t\tau_k e_{\tau_k}(x)e_{\tau_k}(y) \]
Then, we fix an even real-valued function \( \rho \in C^\infty(\mathbb{R}) \) satisfying

\[
\rho(t) = 1 \text{ on } [-1/2, 1/2] \text{ and } \text{supp } \rho \subset (-1, 1),
\]

and define

\[
\tilde{I}_\lambda(\tau) = \frac{1}{\pi} \int_{-\infty}^{\infty} \rho(t) \frac{\sin \lambda t}{t} \cos t\tau \, dt.
\]

Then, by using integral by parts, it is not hard to see

\[
I_\lambda(\tau) - \tilde{I}_\lambda(\tau) = \frac{1}{\pi} \int_{-\infty}^{\infty} (1 - \rho(t)) \frac{\sin \lambda t}{t} \cos t\tau \, dt
\]

\[
= O\left((1 + |\lambda - \tau|)^{-N}\right) \quad \forall \, N.
\]
Thus, by the above inequality, we would have

\[ \left| \int_M (\mathbb{1}_\lambda (P_V)(x, x) - \tilde{\mathbb{1}}_\lambda (P_V)(x, x)) \ dx \right| \]

\[ = \left| \int_M \sum_k (\mathbb{1}_\lambda (\tau_k) - \tilde{\mathbb{1}}_\lambda (\tau_k)) |e_{\tau_k}(x)|^2 \ dx \right| \]

\[ \lesssim \sum_k \int_M (1 + |\lambda - \tau_k|)^{-2n} |e_{\tau_k}(x)|^2 \ dx \lesssim \lambda^{n-1}, \]

if we have

\[ \int_M \sum_{\lambda \leq \tau_k \leq \lambda + 1} |e_{\tau_k}(x)|^2 \ dx \lesssim \lambda^{n-1} \]
If $V \in \mathcal{K}(M) \cap L^{n/2}(M)$, by recent results in Blair-Sire-Sogge(2019), we have the stronger version

$$\sum_{\lambda \leq \tau_k \leq \lambda + 1} |e_{\tau_k}(x)|^2 \lesssim \lambda^{n-1} \text{ uniformly in } x \in M$$

When $V \in \mathcal{K}(M)$, we shall write

$$\int_M \sum_{\lambda \leq \tau_k \leq \lambda + 1} |e_{\tau_k}(x)|^2 \, dx \lesssim \int_M \sum_{\tau_k} \chi(\tau_k - \lambda) |e_{\tau_k}(x)|^2 \, dx$$

$$= \frac{1}{2\pi} \int_M \int_{-\infty}^{\infty} \hat{\chi}(t) e^{-it\lambda} \left(e^{itP_V}(x, x)\right) \, dtdx$$
Duhamel’s principle

To estimate the main term, \( \int_M \tilde{1}_\lambda (P_V)(x, x) \, dx \), note that

\[
(\partial_t^2 + H^0) \left( \cos tP^0 - \cos tP_V \right) f = V(x) \cdot (\cos tP_V) f
\]

with

\[
\left( \frac{d}{dt} \right)^j \left( (\cos tP^0)f - (\cos tP_V)f \right) \bigg|_{t=0} = 0, \quad j = 0, 1,
\]

by Duhamel’s principle we have

\[
(\cos tP^0)f(x) - (\cos tP_V)f(x)
\]

\[
= \int_0^t \left( \frac{\sin(t-s)P^0}{P^0} \left( V \cos(sP_V)f \right) \right)(x) \, ds
\]

\[
= \int_0^t \int \int \sum_j \frac{\sin(t-s)\lambda_j}{\lambda_j} e_j^0(x) e_j^0(z) V(z) \sum_k \cos s\tau_k e_{\tau_k}(z) e_{\tau_k}(y) f(y) \, dzdyds.
\]
Lemma 8.

If $\mu \neq \tau$ we have

$$\int_0^t \frac{\sin(t - s)\mu}{\mu} \cos s\tau \, ds = \frac{\cos t\tau - \cos t\mu}{\mu^2 - \tau^2}.$$  

Similarly,

$$\int_0^t \frac{\sin(t - s)\tau}{\tau} \cos s\tau \, ds = \frac{t \sin t\tau}{2\tau}.$$
Kernels for the main term

By using the above equalities we have

\[
(\tilde{1}_\lambda(P_V) - \tilde{1}_\lambda(P^0))(x, y) = \sum_{j,k} \int_M \frac{\tilde{1}_\lambda(\tau_k) - \tilde{1}_\lambda(\lambda_j)}{\tau_k^2 - \lambda_j^2} e_j^0(x) e_j^0(z) V(z) e_{\tau_k}(z) e_{\tau_k}(y) \, dz,
\]

where we recall that

\[
\tilde{1}_\lambda(\tau) = \frac{1}{\pi} \int_{-\infty}^{\infty} \rho(t) \frac{\sin \lambda t}{t} \cos t\tau \, dt.
\]
Since it is known by previous results that

\[ \tilde{I}_\lambda(P^0)(x,x) = (2\pi)^{-n} \omega_n \lambda^n + O(\lambda^{n-1}), \text{ uniformly in } x \in M. \]

After taking \(dx\) integral it gives us the desired bound
Proposition 9.

If $V \in L^1(M)$ with $V^- \in \mathcal{K}(M)$. Then

$$\left| \sum_{j,k} \int_M \int_M \frac{\tilde{I}_\lambda(\lambda_j) - \tilde{I}_\lambda(\tau_k)}{\lambda_j^2 - \tau_k^2} e_j^0(x)e_j^0(y)V(y)e_{\tau_k}(x)e_{\tau_k}(y) \, dx \, dy \right|$$

$$\leq C_V \| V \|_{L^1(M)} \lambda^{n-\frac{3}{2}},$$

for some constant $C_V$ depending on $V$. 
Some Remarks

• The definition of Kato Class is essentially not used in the proof of integrated Weyl law, for potentials with favorable heat kernel estimates for $e^{-tH_V}$, it is possible to extend our results. For example, the inverse square potential.

• The method also allow us to possibly extend the results of Seeley(1979), which deals with manifolds with boundary.
Thank you for your time