Stability of the Hersch inequality for the first eigenvalue on the 2-sphere and generalizations.

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Based on a joint work with Mickaël Nahon, Iosif Polterovich and Daniel Stern
Shape optimization for Dirichlet eigenvalues

Let $\Omega \subset \mathbb{R}^d$ be a domain in the Euclidean space

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\begin{cases}
\Delta u = \lambda u & \text{on } \Omega, \\
u = 0 & \text{on } \partial \Omega.
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• The eigenvalues form a sequence

\[0 < \lambda_1(\Omega) \leq \lambda_2(\Omega) \leq \cdots \uparrow +\infty\]
Faber-Krahn theorem

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*(Faber, Krahn 1923)* One has

$$
\lambda_1(\Omega)\text{Vol}(\Omega)^{\frac{2}{d}} \geq \lambda_1(B)\text{Vol}(B)^{\frac{2}{d}},
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where $B$ is any ball. Equality iff $\Omega$ is a ball.
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The quantity $\lambda_1(\Omega)\text{Vol}(\Omega)^{\frac{2}{d}}$ can be made arbitrarily large.
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The quantity $\nu_1(\Omega) \text{Vol}(\Omega)^{\frac{2}{d}}$ can be made arbitrarily small.
Stability estimates

\[ A(\Omega) = \inf \{ \frac{\text{Vol}(\Omega - B)}{\text{Vol}(B)} : B \text{ is a ball, } \text{Vol}(B) = \text{Vol}(\Omega) \} \]

**Theorem (Brasco–De Philippis–Velichkov 2015)** There exists \( C_d > 0 \) such that
\[ \lambda_1(\Omega) \frac{\text{Vol}(\Omega)}{2^d} - \lambda_1(B) \frac{\text{Vol}(B)}{2^d} \geq C_d A^2(\Omega). \]

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Let \((M, g)\) be a closed Riemannian surface.

The Laplace-Beltrami operator is defined by

\[ \Delta_g f = -\frac{1}{\sqrt{|g|}} \partial_i (\sqrt{|g|} g^{ij} \partial_j f), \]

where \(g_{ij}\) is the Riemannian metric, \(g^{ij}\) are the components of the matrix inverse to \(g_{ij}\) and \(|g| = \text{det} g\).
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Eigenvalues of the Laplacian

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Set

$$\bar{\lambda}_k(M, g) = \lambda_k(M, g) \text{Area}(M, g).$$
Hersch theorem

Theorem

(Hesch 1970) For any metric $g$ on $S^2$ one has

$$\bar{\lambda}_1(S^2, g) \leq \bar{\lambda}_1(S^2, g_0) = 8\pi,$$

where $g_0$ is a round metric on the sphere. Equality iff $g$ is round.
Stability estimate

Theorem (K.–Nahon–Stern–Polterovich, in prep.) There exists $C > 0$ such that

Remarks:

• $W^{−1, 2} = (W^{1, 2})^∗$ is the borderline norm: there is no stability for $(W^{1, p})^∗, p < 2$;
• the power is optimal;
• unclear how to use the Hessian.
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Geometric optimization of eigenvalues

Consider $\bar{\lambda}_1(M, g)$ as a functional on the space $\mathcal{R}$ of Riemannian metrics on $M$.

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We are interested in the following quantities

$$\Lambda_1(M) = \sup_g \bar{\lambda}_1(M, g);$$
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where $c = [g] = \{ e^\omega g | \omega \in C^\infty(M) \}$ is a fixed conformal class of metrics.
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Korevaar (1993): $\Lambda_k(M) < +\infty.$
Weak formulation

Given a measure $\mu$ on $(M, g)$ define the eigenvalues

$$\lambda_k(M, [g], \mu) = \inf_{E_{k+1} \subset C^\infty(M)} \sup_{u \in E_{k+1} \setminus \{0\}} \frac{\int \lvert du \rvert_g^2 dv_g}{\int u^2 \, d\mu}$$

Has been considered by Grigor'yan–Netrusov–Yau (2004), formally defined by Kokarev (2014) and studied in more detail by Girouard–K.–Lagac´e (2020).
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A measure $\mu$ is called *admissible* if there is a natural compact operator $W^{1,2}(M, g) \to L^2(\mu)$. 
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$$
\Lambda^w_1(M, [g]) = \sup_{\mu \neq 0, \mu \text{ is adm.}} \lambda_1(M, [g], \mu) \mu(M) \geq \Lambda_1(M, [g])
$$
Reformulation of Hersch inequality

- Up to a diffeomorphism, there is a unique conformal structure on $S^2$

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Reformulation of Hersch inequality

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- Similarly, our stability inequality can be formulated as
  \[ \bar{\lambda}_1(g_0) - \lambda_1([g_0], \mu)\mu(\mathbb{S}^2) \geq C \inf_{\Phi \in \text{Conf}(\mathbb{S}^2)} \| \lambda_1([g_0], \mu) \Phi_* \mu - \lambda_1(g_0) d\nu_{g_0} \|^2_{W^{-1,2}(g_0)} \]
Existence and regularity theory

Petrides (2014): For any \( c \) on \( M \) there exists "smooth" metric \( g \in c \) such that \( \overline{\lambda}_1(M, g) = \Lambda_1(M, c) \).


Matthiesen–Siffert (2019): there exists "smooth" metric \( g \) such that \( \overline{\lambda}_1(M, g) = \Lambda_1(M) \).

K.–Stern (2020): \( \Lambda_{w1}(M, c) = \Lambda_1(M, c) \) and any maximal measure is a volume measure of a "smooth" metric. Regularity for \( \Lambda_1(M) \) easily follows.
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Qualitative stability in the conformal class

Theorem (K.–Nahon–Stern–Polterovich, in prep.)
Assume $M \neq S^2$ and $[g]$ be a conformal class on $M$. If $\mu_n$ is a sequence of admissible measures such that $\bar{\lambda}_1(M, [g], \mu_n) \to \Lambda_1(M, [g])$, then $\mu_n$ converge in $W_{-1,2}(M, g)$ to a maximal measure.

Remarks.
• Qualitative stability for $\Lambda_1(M)$ easily follows.
• If $M = S^2$, then there are exceptional $\delta$-sequences.
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• Qualitative stability for \( \Lambda_1(M) \) easily follows.

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General quantitative stability
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**Proposed formulation** (KNPS): Assume $M \neq S^2$ and $[g]$ be a conformal class on $M$. There exist $\delta, C > 0$ such that for any admissible $\mu$ satisfying $\Lambda_1(M, [g]) - \bar{\lambda}_1(M, [g], \mu) < \delta$ there exists a maximal measure $\mu_0$ such that

$$\Lambda_1(M, [g]) - \bar{\lambda}_1(M, [g], \mu) \geq C \| \lambda_1(\mu) \mu - \lambda_1(\mu_0) \mu_0 \|^2_{W^{-1,2}(g)}$$
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- Similar statement for $\Lambda_1(M)$;
General quantitative stability

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- Similar statement for $\Lambda_1(M)$;
- Out of reach in full generality. Missing a priori knowledge on the structure of the set of maximal measures.
Particular examples

- Hersch (1970): $\Lambda_1(S^2) = 8\pi$ and the maximum is achieved on the standard metric on $S^2$. Stable.

- Li–Yau (1982): $\Lambda_1(RP^2) = 12\pi$ and the maximum is achieved on the standard metric on $RP^2$. Stable.
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- Nadirashvili (1996): $\Lambda_1(T^2) = \frac{8\pi^2}{\sqrt{3}}$ and the maximum is achieved on the *flat equilateral torus.*
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- Nadirashvili (1996): $\Lambda_1(\mathbb{T}^2) = \frac{8\pi^2}{\sqrt{3}}$ and the maximum is achieved on the *flat equilateral torus*. Stable.
Examples: continued

  \[ \lambda_1(K) = \overline{\lambda}_1(K, g_{\text{max}}), \]
  where \( g_{\text{max}} \) is the unique maximal metric. It is rotationally symmetric, but not flat.

  \[ \lambda_1(\Sigma_2) = 16\pi. \]
  Bolza surface \( w^2 = z^{18}/27 \).
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Bolza surface \( w^2 = z^5 - z \)
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- El Soufi–Ilias–Ros (1996): Let $g_{a,b}$ be a flat metric on $\mathbb{T}^2$ induced by $\mathbb{R}^2/\mathbb{Z}(1,0) \oplus \mathbb{Z}(a, b)$. 
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  If $a^2 + b^2 = 1$, then $\Lambda_1(\mathbb{T}^2, [g_{a,b}]) = \bar{\lambda}_1(\mathbb{T}^2, g_{a,b})$. 
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  Stable.
Thank you for your attention!
Extremality conditions

Theorem (Nadirashvili 1996; El Soufi, Ilias 2008)

Suppose that \((M, g)\) is extremal for the functional \(\bar{\lambda}_k(M, g)\) in the conformal class \([g]\).

Let \(E_k\) be the corresponding eigenspace. Then there exists a collection \(\Phi = (u_1, \ldots, u_{n+1})\), \(u_i \in E_k\) such that \(\Phi: M \rightarrow \mathbb{R}^{n+1}\) is a map to the unit sphere \(S^n \subset \mathbb{R}^{n+1}\).

The converse holds provided \(\lambda_k - 1(M, g) < \lambda_k(M, g)\).
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The converse holds provided \(\lambda_{k-1}(M, g) < \lambda_k(M, g)\).
Extremality conditions

Theorem (Nadirashvili 1996; El Soufi, Ilias 2008)

Suppose that $(M, g)$ is extremal for the functional $\bar{\lambda}_k(M, g)$ in the space of all metrics and $\lambda_k(M, g) = 2$.

Let $E_k$ be the corresponding eigenspace. Then there exists a collection $\Phi = (u_1, \ldots, u_{n+1})$, $u_i \in E_k$ such that $\Phi: M \to S^n$ is an isometric (branched) immersion to the unit sphere.

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The converse holds provided \(\lambda_{k-1}(M, g) < \lambda_k(M, g)\).
Let $(M, g)$ and $(N, h)$ be Riemannian manifolds and $h$ be a Riemannian metric on $N$. An immersion $\Phi: M \hookrightarrow N$ is called minimal isometric if it is extremal for the volume functional $V(\Phi) = \int_{M} dv_{\Phi^{\ast}h}$ and $\Phi^{\ast}h = g$.

A smooth map $\Phi: M \rightarrow N$ is called harmonic if $\Phi$ is extremal for the energy functional $E_g(\Phi) = \frac{1}{2} \int_{M} |df(x)|^2 dv_g$. 

Harmonic and minimal maps
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Harmonic and minimal maps to $S^n$

- If $N = S^n$ with the standard metric and $\Delta \Phi = \lambda \Phi$, then:
  - $\Phi$ is harmonic;
  - If $\Phi$ is isometric then it is minimal and $\lambda = \dim M$.

Therefore:
- Extremal metrics in conformal class correspond to harmonic maps to $S^n$.
- Extremal metrics in the space of all metrics correspond to (branched) minimal immersions to $S^n$. 
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Maximal metrics for $\lambda_1$: first examples

• Hersch (1970): $\Lambda_1(S^2) = 8\pi$ and the maximum is achieved on the standard metric on $S^2$.

• Li–Yau (1982): $\Lambda_1(\mathbb{R}P^2) = 12\pi$ and the maximum is achieved on the standard metric on $\mathbb{R}P^2$.

• Nadirashvili (1996): $\Lambda_1(T^2) = \frac{8\pi^2}{\sqrt{3}}$ and the maximum is achieved on the flat equilateral torus.
Maximal metrics: $S^2$ and $\mathbb{RP}^2$ revisited

- The eigenfunctions of $S^2 \subset \mathbb{R}^3$ are the restrictions of homogeneous harmonic polynomials $p$ on $\mathbb{R}^3$.
  - Eigenvalue is $\text{deg} p (\text{deg} p + 1)$
    - Degree 1: $x, y, z$
    - Degree 2: $xy, yz, xz, x^2 - y^2, x^2 - z^2$

- $S^2$: the identity map $S^2 \rightarrow S^2$ is an isometric minimal immersion.

- $\mathbb{RP}^2$: Veronese immersion $v: \mathbb{RP}^2 \rightarrow S^4$
  $$v(x, y, z) = (xy, xz, yz, \sqrt{3} (x^2 - y^2), \frac{1}{2} (x^2 + y^2) - z^2)$$
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General quantitative stability estimate

Theorem (KNPS)

Assume that \((M, [g])\) is such that for any harmonic map to \(S^n\) corresponding to a maximal measure there are no non-trivial Jacobi fields. Then the quantitative stability holds.
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Assume that $(M, [g])$ is such that for any harmonic map to $\mathbb{S}^n$ corresponding to a maximal measure there are no non-trivial Jacobi fields. Then the quantitative stability holds.