

Lecture 4: Cauchy sequences, Bolzano-Weierstrass, and the Squeeze theorem

The purpose of this lecture is more modest than the previous ones. It is to state certain conditions under which we are guaranteed that limits of sequences converge.

Definition We say that a sequence of real numbers $\{a_n\}$ is a *Cauchy sequence* provided that for every $\epsilon > 0$, there is a natural number N so that when $n, m \geq N$, we have that $|a_n - a_m| \leq \epsilon$.

Example 1 Let x be a real number and $t_n(x)$ be the n th truncation of its decimal expansion as in Lectures 2 and 3. Then if $n, m \geq N$, we have that $|t_n(x) - t_m(x)| \leq 10^{-N}$, since they share at least the first N places of their decimal expansion. Given any real number $\epsilon > 0$, there is an $N(\epsilon)$ so that $10^{-N(\epsilon)} < \epsilon$. Thus we have shown that the sequence $\{t_n(x)\}$ is a Cauchy sequence.

Example 1 was central in our construction of the real numbers. We got the least upper bound property by associating to each sequence as in Example 1, the real number x which is its limit. The class of Cauchy sequences should be viewed as minor generalization of Example 1 as the proof of the following theorem will indicate.

Theorem 1 Every Cauchy sequence of real numbers converges to a limit.

Proof of Theorem 1 Let $\{a_n\}$ be a Cauchy sequence. For any j , there is a natural number N_j so that whenever $n, m \geq N_j$, we have that $|a_n - a_m| \leq 2^{-j}$. We now consider the sequence $\{b_j\}$ given by

$$b_j = a_{N_j} - 2^{-j}.$$

Notice that for every n larger than N_j , we have that $a_n > b_j$. Thus each b_j serves as a lower bound for elements of the Cauchy sequence $\{a_n\}$ occurring later than N_j . Each element of the sequence $\{b_j\}$ is bounded above by $b_1 + 1$, for the same reason. Thus the sequence $\{b_j\}$ has a least upper bound which we denote by L . We will show that L is the limit of the sequence $\{a_n\}$. Suppose that $n > N_j$. Then

$$|a_n - L| < 2^{-j} + |a_n - b_j| = 2^{-j} + a_n - b_j \leq 3(2^{-j}).$$

For every $\epsilon > 0$ there is $j(\epsilon)$ so that $2^{1-j} < \epsilon$ and we simply take $N(\epsilon)$ to $N_{j(\epsilon)}$.

The idea of the proof of Theorem 1 is that we recover the limit of the Cauchy sequence by taking a related least upper bound. So we can think of the process of finding the limit of the Cauchy sequence as specifying the decimal expansion of the limit, one digit at a time, as this how the least upper bound property worked.

The converse of Theorem 1 is also true.

Theorem 2 Let $\{a_n\}$ be a sequence of real numbers converging to a limit L . Then the sequence $\{a_n\}$ is a Cauchy sequence.

Proof of Theorem 2 Since $\{a_n\}$ converges to L , for every $\epsilon > 0$, there is an $N > 0$ so that when $j > N$, we have

$$|a_j - L| \leq \frac{\epsilon}{2}.$$

(The reason we can get $\frac{\epsilon}{2}$ on the right hand side is that we put $\frac{\epsilon}{2}$ in the role of ϵ in the definition of the limit.) Now if j and k are both more than N , we have $|a_j - L| \leq \frac{\epsilon}{2}$ and $|a_k - L| \leq \frac{\epsilon}{2}$. Combining these using the triangle inequality, we get

$$|a_j - a_k| \leq \epsilon,$$

so that the sequence $\{a_j\}$ is a Cauchy sequence as desired.

Combining Theorems 1 and 2, we see that what we have learned is that Cauchy sequences of real numbers and convergent sequences of real numbers are the same thing. But the advantage of the Cauchy criterion is that to check whether a sequence is Cauchy, we don't need to know the limit in advance.

Example 2 Consider the series (that is, infinite sum)

$$S = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

We may view this as the limit of the sequence of partial sums

$$a_j = \sum_{n=1}^j \frac{1}{n^2}.$$

We can show that the limit converges using Theorem 1 by showing that $\{a_j\}$ is a Cauchy sequence. Observe that if $j, k > N$, we definitely have

$$|a_j - a_k| \leq \sum_{n=N}^{\infty} \frac{1}{n^2}.$$

It may be difficult to get an exact expression for the sum on the right, but it is easy to get an upper bound.

$$\sum_{n=N}^{\infty} \frac{1}{n^2} \leq \sum_{n=N}^{\infty} \frac{1}{n(n-1)} = \sum_{n=N}^{\infty} \frac{1}{n-1} - \frac{1}{n}.$$

The reason we used the slightly wasteful inequality, replacing $\frac{1}{n^2}$ by $\frac{1}{n^2-n}$ is that now the sum on the right telescopes, and we know it is exactly equal to $\frac{1}{N-1}$. To sum up, we have shown that when $j, k > N$, we have

$$|a_j - a_k| \leq \frac{1}{N-1}.$$

Since we can make the right hand side arbitrarily small by taking N sufficiently large, we see that $\{a_j\}$ is a Cauchy sequence. This example gives an indication of the power of the Cauchy criterion. You would not have found it easier to prove that the limit exists if I had told you in advance that the series converges to $\frac{\pi^2}{6}$.

Let $\{a_n\}$ be a sequence of real numbers. Let $\{n_k\}$ be a strictly increasing sequence of natural numbers. We say that $\{a_{n_k}\}$ is a subsequence of $\{a_n\}$. We will now prove an important result which helps us discover convergent sequences in the wild.

Theorem 3(Bolzano-Weierstrass) Let $\{a_n\}$ be a bounded sequence of real numbers. (That is, suppose there is a positive real number B , so that $|a_j| \leq B$ for all j .) Then $\{a_n\}$ has a convergent subsequence.

Proof of Bolzano-Weierstrass All the terms of the sequence live in the interval

$$I_0 = [-B, B].$$

We cut I_0 into two equal halves(which are $[-B, 0]$ and $[0, B]$). At least one of these contains an infinite number of terms of the sequence. We choose a half which contains infinitely many terms and we call it I_1 . Next, we cut I_1 into two halves and choose one containing infinitely many terms, calling it I_2 . We keep going. (At the j th step, we have I_j containing infinitely many terms and we find a half, I_{j+1} which also contains infinitely many terms.) We define the subsequence $\{a_{j_k}\}$ by letting a_{j_k} be the first term of the sequence which follows $a_{j_1}, \dots, a_{j_{k-1}}$ and which is an element of I_j . We claim that $\{a_{j_k}\}$ is a Cauchy sequence. Let's pick $k, l > N$. Then both a_{j_k} and a_{j_l} lie in the interval I_N which has length $\frac{B}{2^{N-1}}$. Thus

$$|a_{j_k} - a_{j_l}| \leq \frac{B}{2^{N-1}}.$$

We can make the right hand side arbitrarily small by making N sufficiently large. Thus we have shown that the subsequence is a Cauchy sequence and hence convergent.

A question you might ask yourselves is: How is the proof of the Bolzano Weierstrass theorem related to decimal expansions?

Our final topic for today's lecture is the Squeeze theorem. It is a result that allows us to show that limits converge by comparing them to limits that we already know converge.

Theorem 4(Squeeze theorem) Given three sequences of real numbers $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$. If we know that $\{a_n\}$ and $\{b_n\}$ both converge to the same limit L and we know that for each n we have

$$a_n \leq c_n \leq b_n,$$

then the sequence $\{c_n\}$ also converges to the limit L .

Proof of Squeeze theorem Fix $\epsilon > 0$. There is $N_1 > 0$ so that when $n > N_1$, we have

$$|a_n - L| \leq \epsilon.$$

There is $N_2 > 0$ so that when $n > N_2$, we have

$$|b_n - L| \leq \epsilon.$$

We pick N to be the larger of N_1 and N_2 . For $n > N$, the two inequalities above, we know that $a_n, b_n \in (L - \epsilon, L + \epsilon)$. But by the inequality

$$a_n \leq c_n \leq b_n,$$

we know that $c_n \in [a_n, b_n]$. Combining the two facts, we see that

$$c_n \in (L - \epsilon, L + \epsilon),$$

so that

$$|c_n - L| \leq \epsilon.$$

Thus the sequence $\{c_n\}$ converges to L as desired.

Example 3 Calculate

$$\lim_{n \rightarrow \infty} \left(1 + \frac{n}{n+1}\right)^{\frac{1}{n}}.$$

The limit above seems a little complicated so we invoke the squeeze theorem. We observe that the inside of the parentheses is between 1 and 2. (Actually it is getting very close to 2 as n gets large. Thus

$$1^{\frac{1}{n}} \leq \left(1 + \frac{n}{n+1}\right)^{\frac{1}{n}} \leq 2^{\frac{1}{n}}.$$

Thus we will know that

$$\lim_{n \rightarrow \infty} \left(1 + \frac{n}{n+1}\right)^{\frac{1}{n}} = 1,$$

provided we can figure out that

$$\lim_{n \rightarrow \infty} 1^{\frac{1}{n}} = 1.$$

and

$$\lim_{n \rightarrow \infty} 2^{\frac{1}{n}} = 1.$$

The first limit is easy since every term of the sequence is 1. It seems to us that the n th roots of two are getting closer to 1, but how do we prove it. Again, it seems like a job for the squeeze theorem. Observe that

$$\left(1 + \frac{1}{n}\right)^n \geq 2,$$

since $1 + \frac{1}{n}$ are the first two terms in the binomial expansion. Thus

$$2^{\frac{1}{n}} \leq 1 + \frac{1}{n}.$$

We know that

$$\lim_{n \rightarrow \infty} 1^{\frac{1}{n}} = 1,$$

and perhaps we also know that

$$\lim_{n \rightarrow \infty} 1 + \frac{1}{n} = 1,$$

since $\frac{1}{n}$ becomes arbitrarily small as n gets large. Thus by the squeeze theorem, we know

$$\lim_{n \rightarrow \infty} 2^{\frac{1}{n}} = 1,$$

and hence

$$\lim_{n \rightarrow \infty} \left(1 + \frac{n}{n+1}\right)^{\frac{1}{n}} = 1.$$

Example 3 is a reasonable illustration of how the squeeze theorem is always used. We might begin with a very complicated limit, but as long as we know the size of the terms concerned, we can compare, using inequalities to a much simpler limit.

As of yet, we have not said anything about infinite limits. We say that a sequence $\{a_n\}$ of positive real numbers converges to infinity if for every $M > 0$, there is an N so that when $n > N$, we have $a_n > M$. Here M takes the role of ϵ . It is measuring how close the sequence gets to infinity. There is a version of the squeeze theorem we can use to show limits go to infinity.

Theorem 5(infinite squeeze theorem) Let $\{a_n\}$ be a sequence of positive real numbers going to infinity. Suppose for every n , we have

$$b_n \geq a_n.$$

Then the sequence $\{b_n\}$ converges to infinity.

Proof of the infinite squeeze theorem For every M , there exists N so that when $n > N$, we have $a_n > M$. But since $b_n \geq a_n$, it is also true that $b_n > M$. Thus $\{b_n\}$ goes to infinity.

Example 4

$$\sum_{n=1}^{\infty} \frac{1}{n} = \infty.$$

We will prove this by comparing each reciprocal to the largest power of two smaller than it. Thus

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \dots > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \dots$$

Combining like terms, we get

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \dots > 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots$$

On the right hand side, we are summing an infinite number of $\frac{1}{2}$'s. Thus the sum is infinite.

Something to think about: Often one shows that the harmonic series diverges by comparing it to the integral of $\frac{1}{x}$ which is a logarithm. Are there any logarithms hiding in Example 4?