Lecture 8: Differentiation: local theory.

Today we'll define the derivative of a function and describe its familiar local theory.

Before doing this we'll introduce a bit of notation, common in some applied fields like the analysis of algorithms, but not often used when discussing single variable calculus. We will do so, because it makes the proofs of the main rules of differentiation, like the product rule and the chain, extremely transparent.

Little-oh and Big-oh notation We say that a function $f(h)$ is $o(h)$ if as $h \longrightarrow 0$ if

$$\lim_{h \longrightarrow 0} \frac{f(h)}{h} = 0.$$

More generally, if $g(h)$ is a continuous increasing function of h with $g(0) = 0$, we say that $f(h)$ is $o(g(h))$ if

$$\lim_{h \longrightarrow 0} \frac{f(h)}{g(|h|)} = 0.$$

We say that f is $O(h)$ as $h \longrightarrow 0$ if there exist $C, \epsilon > 0$ so that for $|h| < \epsilon$, we have

$$|f(h)| \le C|h|.$$

More generally, if $g(h)$ is a continuous increasing function of h with $g(0) = 0$, we say that $f(h)$ is $O(g(h))$ if there exist $C, \epsilon > 0$ so that for $|h| < \epsilon$, we have

$$|f(h)| \le Cg(|h|).$$

Big oh and little oh notation is about describing the size of functions of h asymptotically as $h \longrightarrow 0$. Now we will see how this relates to differentiation. We first give the familiar definition of the derivative.

Definition A function f is differentiable at x if

$$\lim_{h \longrightarrow 0} \frac{f(x+h) - f(x)}{h}$$

exists. We denote this limit by $f'(x)$ or $\frac{d}{dx}(f(x))$.

We can reformulate this using little-oh notation. A function f is differentiable at x if and only if there is a number $f'(x)$ so that

$$f(x+h) = f(x) + hf'(x) + o(h).$$

(Here when we write $+o(h)$ this is shorthand for adding a function which is $o(h)$.) The formula above is called the differential approximation for f. It says that ignoring an $o(h)$ error, the function f is approximated by a linear one with slope $f'(x)$.

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We see immediately from the differential approximation that if f is differentiable at x then
$$f(x + h) = f(x) + O(h).$$

From this it can be show that if f is differentiable at x then f is continuous at x.

Theorem (The product rule) If $f(x)$ and $g(x)$ are functions differentiable at x, then the product $f(x)g(x)$ is differentiable at x and

$$\frac{d}{dx}(f(x)g(x)) = f(x)g'(x) + g(x)f'(x).$$

Proof Since f and g are differentiable at x, we have

$$f(x + h) = f(x) + hf'(x) + o(h),$$

and

$$g(x + h) = g(x) + hg'(x) + o(h).$$

Now we multiply these two equations together.

$$f(x + h)g(x + h) = (f(x) + hf'(x) + o(h))(g(x) + hg'(x) + o(h))$$

$$= f(x)g(x) + h(f(x)g'(x) + g(x)f'(x)) + o(h).$$

Thus the theorem is proved.

Theorem (The chain rule) Suppose that g is differentiable at x and f is differentiable at $g(x)$. Then the function $q(x) = f(g(x))$ is differentiable at x and

$$q'(x) = f'(g(x))g'(x).$$

Proof of the chain rule We calculate

$$\begin{aligned}
&q(x + h) - q(x) \\
&= f(g(x + h)) - f(g(x)) \\
&= [g(x + h) - g(x)]f'(g(x)) + o(g(x + h) - g(x)).
\end{aligned} \tag{1}$$

Here, in the third line, we have just used the differentiability of f at $g(x)$. Now since g is differentiable at x, we have that $g(x + h) - g(x)$ is $O(h)$.

In the third line of equation (1), we have $o(g(x + h) - g(x))$ which is $o(O(h))$ which is $o(h)$.

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Thus rewriting equation (1), we get

$$q(x+h) - q(x)$$
$$= [g(x+h) - g(x)]f'(g(x)) + o(h)$$
$$= [g'(x)h + o(h)]f'(g(x)) + o(h) \tag{2}$$
$$= f'(g(x))g'(x)h + o(h).$$

Here in the third line of equation (2), we have used the differentiability of g at x. Thus we have proved the theorem.

We can go a long way towards building up all of differential calculus using just the product rule and the chain rule (as well as some simpler things like the sum rule.)

Sum rule If $f(x)$ and $g(x)$ are differentiable at x then

$$\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x).$$

Proof

$$(f(x+h) + g(x+h) - f(x) - g(x)) = h(f'(x) + g'(x)) + o(h).$$

Here we have used the commutativity of addition as well as the fact that $o(h)+o(h) = o(h)$.

Power rule for natural numbers Let $n \in \mathbf{N}$. Let $f(x) = x^n$. Then

$$f'(x) = nx^{n-1}.$$

Proof of the power rule for natural numbers Of course, we prove this by induction on n. First we do the base case.

$$(x+h) - x = h = h + o(h).$$

Here we used the fact that 0 is $o(h)$.

Now we do the induction step. We assume that the derivative of x^{n-1} is $n - 1x^{n-2}$. We write

$$f(x) = x^{n-1}x.$$

Now we apply the product rule.

$$f'(x) = (n-1)x^{n-2}x + x^{n-1} = nx^{n-1}.$$

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Quotient rule, version 1 Suppose $f(x), g(x)$ and $\frac{f(x)}{g(x)}$ are differentiable at x and $g(x) \neq 0$, then
$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}.$$

Proof of Quotient rule, version 1 We just write
$$f(x) = \left(\frac{f(x)}{g(x)}\right)g(x),$$

and apply the product rule getting
$$f'(x) = \frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)g(x) + g'(x)\left(\frac{f(x)}{g(x)}\right).$$

We now just solve for $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)$.

The quotient rule is in fact a bit stronger.

Quotient rule, version 2 Let $f(x)$ and $g(x)$ be differentiable at x and let $g(x) \neq 0$. Then $\frac{f(x)}{g(x)}$ is differentiable.

Proof of Quotient rule, version 2

$$\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}$$
$$= \frac{f(x+h)g(x) - f(x)g(x+h)}{g(x)g(x+h)}$$
$$= \frac{f(x)g(x) - f(x)g(x) + (f'(x)g(x) - f(x)g'(x))h + o(h)}{g(x)g(x+h)}$$
$$= \frac{(f'(x)g(x) - f(x)g'(x))h + o(h)}{g(x)(g(x) + O(h))}$$
$$= \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}h + o(h).$$

We can use the chain rule to obtain the inverse rule.

Inverse rule, version 1 Suppose that $f(g(x)) = x$ and g is differentiable at x and f is differentiable at $g(x)$ then
$$f'(g(x)) = \frac{1}{g'(x)}.$$

Proof We just apply the chain rule to $f(g(x)) = x$ and solve for $f'(g(x))$.

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In fact, there is a stronger version of this, guaranteeing the differentiability of the inverse f at $g(x)$ if $g'(x) \neq 0$ and in fact guaranteeing that f exists under that condition.

An application is that this allows us to differentiate rational powers. We define

$$x^{\frac{1}{n}} = l.u.b\{y : y^n < x\}.$$

It is easy to see that

$$(x^{\frac{1}{n}})^n = x.$$

Differentiating both sides, the left using the chain rule, we get

$$1 = \frac{d}{dx}(x^{\frac{1}{n}})^n$$

$$= n(x^{\frac{1}{n}})^{n-1}\frac{d}{dx}(x^{\frac{1}{n}}).$$

We solve obtaining

$$\frac{d}{dx}(x^{\frac{1}{n}}) = \frac{1}{n}x^{-\frac{n-1}{n}}.$$

We can actually define irrational powers as limits of rationals but we delay this to our study of exponential functions. As it is, we can differentiate all algebraic functions and this is a course in which we do "late transcendentals."