Lecture 9: The mean value theorem

Today, we'll state and prove the mean value theorem and describe other ways in which derivatives of functions give us global information about their behavior.

Let f be a real valued function on an interval $[a, b]$. Let c be a point in the interior of $[a, b]$. That is, $c \in (a, b)$. We say that f has a local maximum (respectively local minimum) at c if there is some $\epsilon > 0$ so that $f(c) \geq f(x)$ (respectively $f(c) \leq f(x)$) for every $x \in (c - \epsilon, c + \epsilon)$.

Lemma Let f be a real valued function on $[a, b]$, differentiable at the point c of the interior of $[a, b]$. Suppose that f has a local maximum or local minimum at c. Then

$$f'(c) = 0.$$

Proof of Lemma Since f is differentiable at c, we have that

$$f(x) = f(c) + f'(c)(x - c) + o(|x - c|),$$

as $x - c \longrightarrow 0$. Suppose that $f'(c) \neq 0$. From the definition of o, we have that there is some $\delta > 0$ so that

$$|f(x) - f(c) - f'(c)(x - c)| \leq \frac{|f'(c)||x - c|}{2},$$

whenever

$$|x - c| < \delta.$$

(This is true since indeed we can choose δ to bound by $\epsilon|x - c|$ for any $\epsilon > 0$.) Thus whenever $|x - c| < \delta$, the sign of $f(x) - f(c)$ is the same as the sign of $f'(c)(x - c)$. This sign changes depending on whether $x - c$ is positive or negative. But this contradicts $f(c)$ being either a local maximum or minimum. Thus our initial assumption was false and we have $f'(c) = 0$ as desired.

In high school calculus, this lemma is often used for solving optimization problems. Suppose we have a function f which is continuous on $[a, b]$ and differentiable at every point in the interior of $[a, b]$. Then from the extreme value theorem, we know the function achieves a maximum on $[a, b]$. One possibility is that the maximum is at a or at b. If this is not the case, then the maximum must be at a point where $f'(c) = 0$. Instead, we shall use the Lemma to prove the Mean Value theorem.

Rolle's theorem Let $f(x)$ be a function which is continuous on the closed interval $[a, b]$ and differentiable on every point of the interior of $[a, b]$. Suppose that $f(a) = f(b)$. Then there is a point $c \in [a, b]$ where $f'(c) = 0$.

Proof of Rolle's theorem By the extreme value theorem, f achieves its maximum on $[a, b]$. By applying the extreme value theorem to $-f$, we see that f also achieves its minimum on $[a, b]$. By hypothesis, if both the maximum and minimum are achieved on the boundary, then the maximum and minimum are the same and thus the function is constant. A constant function has zero derivative everywhere. If f is not constant, then f has either a local minimum or a local maximum in the interior. By the Lemma, the derivative at the local maximum or minimum must be zero.

Mean Value Theorem Let $f(x)$ be a function which is continuous on the closed interval $[a, b]$ and which is differentiable at every point of (a, b). Then there is a point $c \in (a, b)$ so that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Proof of Mean Value Theorem Replace $f(x)$ by

$$g(x) = f(x) - \frac{(f(b) - f(a))(x - a)}{b - a}.$$

Observe that $g(a) = f(a)$ and $g(b) = f(b) - (f(b) - f(a)) = f(a)$. Further g has the same continuity and differentiability properties as f since

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}.$$

Thus we may apply Rolle's theorem to g finding $c \in (a, b)$, where $g'(c) = 0$. We immediately conclude that

$$f'(c) = \frac{f(b) - f(a)}{b - a},$$

proving the theorem.

We can use the Mean value theorem to establish some of our standard ideas about the meaning of the derivative as well as our standard tests for determining whether a critical point, a point c in the interior of the domain of a function f, where $f'(c) = 0$, is a local maximum or a local minimum.

Proposition Suppose a function f is continuous on the interval $[a, b]$ and differentiable at every point of the interior (a, b). Suppose that $f'(x) > 0$ for every $x \in (a, b)$ then $f(x)$ is strictly increasing on $[a, b]$. (That is for every $x, y \in [a, b]$ if $x < y$ then $f(x) < f(y)$.

Proof of Proposition Given $x, y \in [a, b]$ with $x < y$, we have that f satisfies the hypotheses of the mean value theorem on $[x, y]$. Thus there is $c \in (x, y)$ so that

$$f(y) - f(x) = f'(c)(y - x).$$

Since we know that $f'(c) > 0$, we conclude that

$$f(y) - f(x) > 0,$$

or in other words

$$f(y) > f(x).$$

Thus f is increasing.

Theorem(First derivative test) Let f be a function continuous on $[a, b]$ and differentiable on (a, b). Let c be a point of (a, b) where $f'(c) = 0$. Suppose there is some $\delta > 0$ so that for every $x \in (c-\delta, c)$, we have that $f'(x) > 0$ and for every $x \in (c, c+\delta)$, we have that $f'(x) < 0$, then f has a local maximum at c.

Proof By choosing δ sufficiently small, we arrange that $(c - \delta, c + \delta) \subset (a, b)$. Thus, we may apply the previous Proposition to f on $[c - \delta, c]$ concluding that $f(c) > f(x)$ for any $x \in (c-\delta, c]$. Next, we apply the proposition to $-f$ on the interval $[c, c+\delta]$, concluding that $-f(x) > -f(c)$ for any $x \in (c, c + \delta]$. Multiplying the inequality b -1, we see this is the same as $f(c) > f(x)$. Thus f achieves its maximum on $[c - \delta, c + \delta]$ at c. In other words, f has a local maximum at c.

Theorem(Second derivative test) Let f be a function continuous on $[a, b]$ and differentiable on (a, b). Let c be a point of (a, b) where $f'(c) = 0$. Suppose the derivative $f'(x)$ is differentiable at c and that $f''(c) < 0$. Then f has a local maximum at c.

Proof of the second derivative test Since $f'(c) = 0$, we have that

$$f'(x) = f''(c)(x - c) + o(x - c),$$

as $x \longrightarrow c$. In particular, there is $\delta > 0$ so that

$$|f'(x) - f''(c)(x - c)| < \frac{1}{2}|f''(c)||x - c|.$$

(This is true since indeed we can choose δ to bound by $\epsilon|x - c|$ for any $\epsilon > 0$.) Thus for any $x \in (c - \delta, c)$, we have that $f'(x) > 0$, while for any $x \in (c, c + \delta)$ we have $f'(x) < 0$. Thus we may apply the first derivative test to conclude that f has a local maximum at c.