Problem Set 8 Solutions

5.1.1. Since \(1 - x^2 = (1 + x)(1 - x)\) and \(1/\sqrt{1 + x} \leq 1/\sqrt{1 - x}\) for \(0 \leq x < 1\), we see that \(1/\sqrt{1 - x^2} \leq 1/\sqrt{1 - a}\) for \(0 \leq x < 1\). So

\[
\pi = 2 \int_0^1 \frac{dx}{\sqrt{1 - x^2}} = 2 \lim_{a \to 1} \int_0^a \frac{dx}{\sqrt{1 - x^2}} \leq 2 \lim_{a \to 1} \int_0^a \frac{dx}{\sqrt{1 - x}} = 4 \lim_{a \to 1} (\sqrt{1 - 0} - \sqrt{1 - a}) = 4.
\]

In particular, \(\pi\) is finite.

5.1.2. We have \(y(x) = \int_{\pi/6}^x \sqrt{\sec^2 t - 1} dt\). By fundamental theorem of calculus we have \(y'(x) = \sqrt{\sec^2 x - 1}\). Now by definition of arclength:

\[
\text{arclength} = \int_{\pi/6}^{\pi/3} \sqrt{1 + (y'(x))^2} \, dx = \int_{\pi/6}^{\pi/3} \sqrt{1 + \sec^2 x - 1} \, dx = \int_{\pi/6}^{\pi/3} \sec x \, dx = \log(\sec x + \tan x)\bigg|_{\pi/6}^{\pi/3} = \log(2 + \sqrt{3}) - \log(2 + \sqrt{3}) = -\frac{\log(3)}{2} + \log(2 + \sqrt{3}) \approx 0.76765175259076186292.
\]

5.2.1. Consider parallelogram \(ABCD\) such that \(AB//CD\) and \(AD//BC\). We want to show that

\[
|AB|^2 + |AD|^2 + |BC|^2 + |CD|^2 = |AC|^2 + |BD|^2.
\]

Let’s denote \(\vec{a} = \vec{AB}, \vec{b} = \vec{AD}\). By definition of parallelogram,

\[
\vec{BC} = \vec{a}, \vec{BD} = \vec{b}, \vec{AC} = \vec{AB} + \vec{BC} = \vec{a} + \vec{b}, \vec{BD} = \vec{BC} + \vec{CD} = \vec{b} - \vec{a}.
\]

So

\[
|AB|^2 + |AD|^2 + |BC|^2 + |CD|^2 = 2(|\vec{a}|^2 + |\vec{b}|^2) = 2(\vec{a} \cdot \vec{a} + \vec{b} \cdot \vec{b}),
\]

\[
|AC|^2 + |BD|^2 = (\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b}) + (\vec{b} - \vec{a}) \cdot (\vec{b} - \vec{a}) = 2(\vec{a} \cdot \vec{a} + \vec{b} \cdot \vec{b}).
\]

5.2.2. By Theorem 2.3.2 in the notes,

\[
e^x e^y = \left( \sum_{k=0}^{\infty} \frac{x^k}{k!} \right) \left( \sum_{l=0}^{\infty} \frac{y^l}{l!} \right) = \sum_{n=0}^{\infty} \sum_{k+l=n} \frac{x^k y^l}{k! l!}
\]

because the two series in the middle converge absolutely. (Technically that theorem only applies to real \(x, y\); the complex version is an easy corollary, by considering real and imaginary parts separately.) Note that

\[
\sum_{k+l=n} \frac{x^k y^l}{k! l!} = \sum_{k+l=n} \frac{1}{(k+l)!} \frac{(k+l)!}{k! l!} x^k y^l = \frac{1}{n!} \sum_{k+l=n} \binom{k+l}{k} x^k y^l = \frac{(x+y)^n}{n!}
\]
by the binomial theorem; hence
\[ e^x e^y = \sum_{n=0}^{\infty} \frac{(x + y)^n}{n!} = e^{x+y}, \]
as desired.

5.3.1. Suppose that \( i_1, i_2, ..., i_n, ... \) is another ordering of the natural numbers. Since \( \sum_{j=1}^{\infty} a_j \) is absolutely convergent, for any \( \epsilon > 0 \), \( \exists N > 0 \) such that \( \sum_{j=N}^{\infty} |a_j| < \epsilon \). Suppose that \( i_{j_t} = t \) for \( t = 1, 2, ..., N \). Set \( M = \max\{j_1, ..., j_N\} + 1 \). Then \( \sum_{j=M}^{\infty} |a_{i_j}| < \sum_{j=N}^{\infty} |a_j| < \epsilon \). So \( \sum_{j=1}^{\infty} a_{i_j} \) is also absolutely convergent. Moreover, suppose that \( s = \sum_{j=1}^{\infty} a_{i_j} \). Then
\[
\sum_{j=1}^{\infty} a_{i_j} = \sum_{j=1}^{M-1} a_{i_j} + \sum_{j=M}^{\infty} a_{i_j}
\]
\[
= \sum_{j=1}^{N} a_j + \sum_{1 \leq j \leq M-1; j \not\in \{j_t\}}^{N} a_{i_j} + \sum_{j=M}^{\infty} a_{i_j}
\]
\[
= s - \sum_{j=N+1}^{\infty} a_j + \sum_{1 \leq j \leq M-1; j \not\in \{j_t\}}^{N} a_{i_j} + \sum_{j=M}^{\infty} a_{i_j}
\]
\[
\leq s + \sum_{j=N}^{\infty} |a_j| + \sum_{j=N}^{\infty} |a_j| + \sum_{j=M}^{\infty} |a_{i_j}|
\]
\[
= s + 3\epsilon.
\]
Since \( \epsilon \) is arbitrary, we have \( \sum_{j=1}^{\infty} a_{i_j} \leq s \). Since \( 1, 2, ..., n, ... \) is also a permutation of \( i_1, i_2, ..., i_n, ... \), by the same argument we have \( s \leq \sum_{j=1}^{\infty} a_{i_j} \). Hence \( \sum_{j=1}^{\infty} a_{i_j} = s \).