Lecture 14 The Riemann integral defined

Our goal for today is to begin work on integration. In particular, we would like to define \( \int_{a}^{b} f(x) \, dx \), the definite Riemann integral of a function \( f \) on the interval \([a, b]\). Here \( f \) should be, at least, defined and bounded on \([a, b]\).

Informally, the meaning we would like to assign to \( \int_{a}^{b} f(x) \, dx \) is area under the curve \( y = f(x) \) between the vertical lines \( x = a \) and \( x = b \). But we’ll have to come to terms with understanding what that means, and having at least some idea about which curves have a well defined area under them.

Classically, we understand what is the area of a rectangle and not much else. (Parallelograms are rectangles with the same triangle added and subtracted. Triangles are half-parallelograms. The areas of all these objects are built up from area of a rectangle.) Our idea will be that we will study certain unions of disjoint rectangles contained in the region under the curve, whose areas we will call lower Riemann sums, and we will study unions of disjoint rectangles covering the region, whose areas we will call upper Riemann sums, and our integral will be defined when we can squeeze the area of the region tightly between the upper and lower sums. Warning: Being able to do this will put some restrictions on \( f \).

Given an interval \([a, b]\), a partition \( P \) of \([a, b]\) is a set of points \( \{x_0, \ldots, x_n\} \) so that
\[
x_0 = a < x_1 < x_2 \ldots < x_{n-1} < x_n = b.
\]
We say a partition \( Q \) refines the partition \( P \) provided that
\[
P \subset Q,
\]
that is provided every point of \( P \) is also a point of \( Q \).

Given a partition \( P \) of \([a, b]\) and \( f \) a bounded function on \([a, b]\), we define
\[
U_P(f) = \sum_{j=1}^{n} \text{l.u.b.} \{f(x) : x_{j-1} \leq x \leq x_j\} (x_j - x_{j-1}),
\]
the Riemann upper sum of \( f \) with respect to the partition \( P \).

Given a set \( A \) of real numbers bounded below, we define its g.l.b. (greatest lower bound) by
\[
g.l.b.(A) = -l.u.b.(-A),
\]
where \(-A\) is the set of negatives of elements of \( A \). If \( A \) is bounded below then \( g.l.b(A) \) is defined because the negative of a lower bound for \( A \) is an upper bound for \(-A\).
Now, we define
\[ L_P(f) = \sum_{j=1}^{n} \text{g.l.b.}\{f(x) : x_{j-1} \leq x \leq x_j\}(x_j - x_{j-1}), \]
the Riemann lower sum of \( f \) with respect to the partition \( P \).

We record some facts about Riemann upper and lower sums.

Claim Let \( P = \{x_0, x_1, \ldots, x_n\} \) be a partition of \([a, b]\) and let \( Q \) be a partition which refines \( P \) then for any bounded \( f \) defined on \([a, b]\), we have
\[ L_P(f) \leq L_Q(f) \leq U_Q(f) \leq U_P(f). \]

Proof of claim We observe that for every pair of adjacent points of \( P \), namely \( x_{j-1}, x_j \), the subset \( Q_{[x_{j-1}, x_j]} \) consisting of points in \( Q \) contained in \([x_{j-1}, x_j]\) is a partition of \([x_{j-1}, x_j]\). It suffices to show that
\[ \text{g.l.b.}\{f(x) : x_{j-1} \leq x \leq x_j\}(x_j - x_{j-1}) \leq \text{g.l.b.}\{f(x) : x_{j-1} \leq x \leq x_j\}(x_j - x_{j-1}) \]
\[ \leq \text{l.u.b.}\{f(x) : x_{j-1} \leq x \leq x_j\}(x_j - x_{j-1}). \]
This is true because the g.l.b.’s in the definition of \( L_Q_{[x_{j-1}, x_j]} \) are all larger than or equal to the g.l.b. on all of \([x_{j-1}, x_j]\) which in turn are smaller than or equal to the respective l.u.b.’s which are smaller than or equal to the l.u.b. on all of \([x_{j-1}, x_j]\). Now we just sum our inequalities over \( j \) to obtain the desired inequalities.

Corollary of claim Let \( P \) and \( Q \) be any partitions of \([a, b]\) then for any bounded \( f \) on \([a, b]\),
\[ L_P(f) \leq U_Q(f). \]

Proof of corollary Clearly \( P \cup Q \) refines both \( P \) and \( Q \). We simply use the claim to show that
\[ L_P(f) \leq L_{P \cup Q}(f) \leq U_Q(f). \]

Thus we have obtained that the set of all lower Riemann sums of a bounded function on \([a, b]\) are bounded above, and we denote
\[ \text{l.u.b.}\{L_P(f)\} = I_{l, [a, b]}(f), \]
where the l.u.b. is taken over all partitions of \([a, b]\). We call \( I_{l, [a, b]}(f) \) the lower integral of \( f \) on \([a, b]\).
Similarly the upper sums are all bounded below. We denote
g.l.b. \{U_p(f)\} = I_{u,[a,b]}(f),
where the g.l.b. is taken over all partitions of \([a,b]\). We call \(I_{u,[a,b]}(f)\) the upper integral of \(f\) on \([a,b]\).

When these two numbers \(I_{l,[a,b]}(f)\) and \(I_{u,[a,b]}(f)\) are equal, we say that \(f\) is Riemann integrable on \([a,b]\) and we call this common number
\[
\int_a^b f(x)\,dx.
\]

**Warning example**

Let \(f(x)\) be defined on \([0,1]\) by
\[
f(x) = \begin{cases} 
1 & \text{if } x \in \mathbb{Q}; \\
0 & \text{if } x \notin \mathbb{Q}.
\end{cases}
\]

It is easy to see that any upper sum of \(f\) on \([0,1]\) is 1 and any lower sum is 0. The function \(f\) is not Riemann integrable. There are more sophisticated integrals that can handle this \(f\) [I’m looking at you, Lebesgue!!] but no system of integration will work on any function.

We record some basic properties of Riemann integration:

**Theorem** Let \(f, g\) be Riemann integrable on \([a,b]\) and \(c_1, c_2, c, k \neq 0\) be numbers.

(i) \[
\int_a^b c_1 f + c_2 g = c_1 \int_a^b f + c_2 \int_a^b g
\]

(ii) \[
\int_a^b f(x)\,dx = \int_{a+c}^{b+c} f(x-c)\,dx.
\]

(iii) \[
\int_a^b f(x)\,dx = \frac{1}{k} \int_{ka}^{kb} f\left(\frac{x}{k}\right)\,dx.
\]

If for every \(x \in [a,b]\), we have \(g(x) \leq f(x)\) then

(iv) \[
\int_a^b g(x)\,dx \leq \int_a^b f(x)\,dx
\]

If \(c \in [a,b]\),

(v) \[
\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx.
\]
We proceed to prove the parts of the theorem.

**Proof of (i)** We let \( \mathcal{P} \) be any partition of \([a, b]\). We restrict to the case that \( c_1, c_2 \) are nonnegative. We see that

\[
U_{\mathcal{P}}(c_1f + c_2g) \leq c_1U_{\mathcal{P}}(f) + c_2U_{\mathcal{P}}(g),
\]

since the maximum (or near maximum) in every subinterval of the partition may occur at different points for \( f \) and \( g \). Similarly

\[
c_1L_{\mathcal{P}}(f) + c_2L_{\mathcal{P}}g \leq L_{\mathcal{P}}(c_1f + c_2g).
\]

Taking respectively l.u.b and g.l.b., we get

\[
I_u, [a, b] (c_1f + c_2g) \leq c_1 \int_a^b f + c_2 \int_a^b g,
\]

and

\[
I_l, [a, b] (c_1f + c_2g) \geq c_1 \int_a^b f + c_2 \int_a^b g.
\]

Since

\[
I_l, [a, b] (c_1f + c_2g) \leq I_u, [a, b] (c_1f + c_2g),
\]

we have shown that \( c_1f + c_2g \) is Riemann integrable and that (i) holds. To get the full power of (i), we must consider negative \( c_1 \) and \( c_2 \). It is enough to show that if \( f \) is integrable on \([a, b]\) then so is \(-f\). We see immediately that

\[
I_{u, [a, b]}(f) = -I_{l, [a, b]}(-f),
\]

and

\[
I_{l, [a, b]}(f) = -I_{u, [a, b]}(-f).
\]

Thus \(-f\) is integrable with integral \(-\int_a^b f\).

**Proof of (ii)** We see that any partition \( \mathcal{P} \) of \([a, b]\) can be transformed to a partition \( \mathcal{P} + c \) of \([a + c, b + c]\) (and vice versa) and we see that

\[
U_{\mathcal{P}}(f(x)) = U_{\mathcal{P} + c}(f(x - c)),
\]

and

\[
L_{\mathcal{P}}(f(x)) = L_{\mathcal{P} + c}(f(x - c)).
\]

**Proof of (iii)** Similarly, We see that any partition \( \mathcal{P} \) of \([a, b]\) can be transformed to a partition \( k\mathcal{P} \) of \([ka, kb]\) (and vice versa) and we see that

\[
U_{\mathcal{P}}(f(x)) = \frac{1}{k}U_{k\mathcal{P}}(f(x/k)).
\]
and
\[ L_P(f(x)) = \frac{1}{k} L_{kP}(f(\frac{x}{k})). \]

Proof of (iv) We see that for any partition \( P \) of \([a, b] \),
\[ U_P(g(x)) \leq U_P(f(x)). \]
It suffices to take g.l.b of both sides.

Proof of (v) We simply observe that any \( P \) which is a partition for \([a, b] \) can be refined to a union of a partition \( P_1 \) of \([a, c] \) together with a partition \( P_2 \) of \([c, b] \) simply by adding the point \( c \). We conclude
\[ L_P(f) \leq L_{P_1}(f) + L_{P_2}(f) \leq U_{P_1}(f) + U_{P_2}(f) \leq U_P(f). \]