Lecture 25: Analytic functions

Today we will explore the calculus of complex valued functions of a complex number. On your last problem set, you have frequently met complex-valued functions $f(t)$ of a real variable $t$. Complex numbers have real and imaginary parts so we can write

$$ f(t) = u(t) + iv(t). $$

Then, if the derivative exists

$$ f'(t) = u'(t) + iv'(t), $$

which is consistent with

$$ f'(t) = \lim_{h \to 0} \frac{f(t + h) - f(t)}{h}. $$

We can take any differentiable functions of one real variable $u(t)$ and $v(t)$ and combine them as $u(t) + iv(t)$ and obtain a complex valued function of a real variable which is differentiable.

When it comes to function of a complex variable, something more special is going on. When you think of “elementary functions”, those really formulaic functions that we meet in calculus, all of them can be applied to complex numbers. To wit:

$$ e^{x+iy} = e^x e^{iy} = e^x \cos y + ie^x \sin y. $$

We will take $e^z$ with $z$ a complex variable as an example throughout this lecture. Further,

$$ \log(x + iy) = \log(\sqrt{x^2 + y^2}) + \log(\frac{x + iy}{\sqrt{x^2 + y^2}}) = \log(\sqrt{x^2 + y^2}) + i \arcsin(\frac{y}{\sqrt{x^2 + y^2}}). $$

Even trigonometric functions can be applied to a complex variable $z$.

$$ \sin z = \frac{e^{iz} - e^{-iz}}{2i}. $$

Often sin applied to an imaginary argument is referred to as hyperbolic sin.

We will use

$$ f(z) = f(x + iy) = e^{x+iy} = e^x \cos y + ie^x \sin y, $$

as our main example.

A complex valued function $f(z)$ has a real and imaginary part

$$ f(x + iy) = u(x, y) + iv(x, y), $$

each of which is a real function of two real variables. In the case of $e^z$, we have

$$ u(x, y) = e^x \cos y, $$

1
\[ v(x, y) = e^x \sin y. \]

We will briefly discuss the differentiation theory of real functions of two variables. You can view this as preview of Math 1c.

Recall that a function of one variable \( f(x) \) is differentiable at \( x \) exactly when there is a number \( f'(x) \) so that
\[
f(x + h) = f(x) + f'(x)h + o(h).
\]

Given a function \( f(x, y) \) of two variables, we can leave \( y \) fixed and view it purely as a function of \( x \) and then take the derivative. This derivative \( \frac{\partial f}{\partial x} \) is called the partial derivative in \( x \) and is defined by
\[
f(x + h, y) = f(x, y) + \frac{\partial f}{\partial x}(x, y)h + o(h).
\]

Similarly \( \frac{\partial f}{\partial y}(x, y) \) is defined by
\[
f(x, y + k) = f(x, y) + \frac{\partial f}{\partial y}(x, y)k + o(k).
\]

If \( \frac{\partial f}{\partial x} \) and \( \frac{\partial f}{\partial y} \) are defined and continuous at all points near \( (x, y) \) then something stronger is true. Namely
\[
f(x + h, y + k)
\begin{align*}
&= f(x + h, y) + \frac{\partial f}{\partial y}(x + h, y)k + o(k) \\
&= f(x, y) + \frac{\partial f}{\partial x}(x, y)h + \frac{\partial f}{\partial y}(x + h, y)k + o(h) + o(k) \\
&= f(x, y) + \frac{\partial f}{\partial x}(x, y)h + \frac{\partial f}{\partial y}(x, y)k + o(1)k + o(h) + o(k) \\
&= f(x, y) + \frac{\partial f}{\partial x}(x, y)h + \frac{\partial f}{\partial y}(x, y)k + o(h) + o(k)
\end{align*}
\]

Note that the equality between the third and fourth line used the continuity of \( \frac{\partial f}{\partial y} \). The equality between the first and fifth line is usually taken as the definition of differentiability and is the two-variable differential approximation.

Leibniz introduced shorthand for differential approximations. The one variable differential approximation for \( f(x) \) a differentiable function of one variable becomes
\[
df = \frac{df}{dx} dx.
\]
Shorthand for the differential approximation of \( f(x, y) \) a differentiable function of two variables is
\[
df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy.
\]

Now we are ready to see what these differentials are in the case of our central example
\[
f(x, y) = e^x \cos y + ie^x \sin y.
\]

We calculate
\[
df = (e^x \cos y + ie^x \sin y)dx + (-e^x \sin y + ie^x \cos y)dy
\]
\[
= e^x \cos y(dx + idy) + e^x \sin y(idx - dy)
\]
\[
= (e^x \cos y + ie^x \sin y)(dx + idy)
\]
\[
= e^z dz
\]

!!! Here we use \( z = x + iy \) and \( dz = dx + idy \) and what we have seen is that with \( f(x, y) = e^z \), we have the differential approximation \( df = e^z dz \).

This motivates a definition.

**Definition** A complex valued function of a complex variable \( f(z) \) which is differentiable at \( z \) as a function of two variables is **analytic** at \( z \) if \( \frac{df}{dz} \) cleans up, that is if \( df(z) \) is a complex multiple of \( dz \).

Analyticity is a very special condition. It is not the case that if we write
\[
f(z) = u(x, y) + iv(x, y),
\]
for any nice functions \( u \) and \( v \) that we please, we will end up with an analytic function. Let’s see what conditions analyticity puts on \( u \) and \( v \).

\[
\frac{df}{dz} = du + idv
\]
\[
= \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + i(\frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy)
\]
\[
= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)dy.
\]

Now it is clear that \( \frac{df}{dz} \) cleans up means, that
\[
\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y} = i(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}).
\]

Breaking this equation into real and imaginary parts, we get
\[
\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = 0,
\]
\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} = 0.
\]
\[
\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},
\]
\[
\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}.
\]

These equations, giving the conditions on \( u \) and \( v \) for \( u + iv \) to be analytic are called the Cauchy-Riemann equations.

There are various alternative interpretations of the condition of analyticity. In particular \( f(z) \) is analytic at \( z \) if
\[
\lim_{h \to 0} \frac{f(z + h) - f(z)}{h}
\]
exists as a complex limit, (that is, with \( h \) running over the complex numbers.) The reason that this is so much more restrictive than ordinary differentiation, is , again the role played by complex multiplication.

To have a complete picture of analytic functions of a complex variable, it is not enough just to be able to differentiate. We should also be able to integrate. Next time, we’ll worry about when it makes sense to integrate an expression of the form \( f(z)dz \) with \( f \) an analytic function.

Today we leave that as a mystery. One clue is that if we write
\[
f = u + iv,
\]
then
\[
f(z)dz = udx - vdy + i(vdx +udy).
\]
A perhaps more satisfying clue is that if \( F \) is a complex antiderivative for \( f \), we have that
\[
dF = f(z)dz.
\]

For next time, we leave:

**Question:** When does \( f(z) \) analytic have a complex antiderivative?