Lecture 27 Taylor series of analytic functions

Today our goal is to show all analytic functions are given by convergent power series.

Last time we saw:
If \( \alpha \) is a closed curve contained in a rectangle \( R \) where \( f \) is analytic then
\[
\int_{\alpha} f(z) \, dz = 0.
\]

We also saw that if \( \alpha \) was a circle centered at 0
\[
\int_{\alpha} \frac{dz}{z} = 2\pi i.
\]

In fact, as long as the circle surrounds 0,
\[
\int_{\alpha} \frac{dz}{z} = 2\pi i.
\]

It is just because
\[
d\log z = \frac{dz}{z}.
\]

Now we want to combine these facts. We'll begin with a function analytic in a small rectangle \( R \) containing 0 in its interior. We'll let \( \alpha \) be a circle of some radius \( r \) centered at 0 but with \( r \) small enough so that the circle is contained in the rectangle \( R \). Finally for convenience, we'll also assume that \( f'(z) \) is analytic in \( R \). (But we're only assuming we have two derivatives of \( f \), something you can check in finite time for any elementary function of calculus, where defined.)

Fact
\[
0 = \frac{1}{2\pi i} \int_{\alpha} \frac{(f(z) - f(0)) \, dz}{z}.
\]

If the fact is true, then we get
\[
f(0) = \frac{1}{2\pi i} \int_{\alpha} \frac{f(0) \, dz}{z} = \frac{1}{2\pi i} \int_{\alpha} f(z) \, dz.
\]

To see the fact, we need that \( \frac{f(z) - f(0)}{z} \) is analytic on \( R \). Clearly this is OK at every point but \( z = 0 \). Clearly, the function is continuous at 0, limiting to \( f'(0) \). We just need to check it has a derivative there. We calculate
\[
\lim_{h \to 0} \frac{f(h) - f(0)}{h} - f'(0) = \lim_{h \to 0} \frac{f(h) - f(0) - hf'(0)}{h^2}
\]
\[
= \lim_{h \to 0} \frac{h^2 f''(0) + o(h^2)}{h^2}
\]
\[
= \frac{1}{2} f''(0).
\]
So we conclude

\[ f(0) = \int_{\alpha} \frac{f(z)dz}{z}. \]

Now we apply the same argument for \( w \) near 0. (In particular, on the inside of the circle \( \alpha \).) We get

\[ f(w) = \frac{1}{2\pi i} \int_{\alpha} \frac{f(z)dz}{z-w}. \]

What’s amazing about this formula is that we’ve moved the entire dependence of the function on \( w \) inside this integral. In fact, all of the behavior of \( f \) is determined by its values on the circle \( \alpha \).

Now \( f(z) \) is continuous on \( \alpha \) so by the extreme value theorem, it is actually bounded. That is

\[ |f(z)| \leq M, \]

for some real number \( M \) when \( z \) is on the circle \( \alpha \).

Moreover we can calculate derivatives by differentiating under the integral sign. (Why? Compare to the solution of Hwk 8, problem 5.)

We get

\[ f'(w) = \frac{1}{2\pi i} \int_{\alpha} \frac{f(z)dz}{(z-w)^2}, \]

\[ f''(w) = \frac{1}{2\pi i} \int_{\alpha} \frac{2f(z)dz}{(z-w)^3}, \]

\[ f^{(n)}(w) = \frac{1}{2\pi i} \int_{\alpha} \frac{n!f(z)dz}{(z-w)^{n+1}}. \]

Thus

\[ |f^{(n)}(0)| = \left| \frac{1}{2\pi i} \int_{\alpha} \frac{n!f(z)dz}{(z)^{n+1}} \right| \]

\[ \leq \frac{1}{2\pi} \int_{\alpha} \frac{|n!f(z)|dz}{|z|^{n+1}} \]

\[ \leq \frac{M n!}{r^n}. \]

Thus if we write out the Taylor series of \( f \) at 0, we get

\[ \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n. \]
The $n$th coefficient is bounded in absolute value by $\frac{M}{r^n}$. These are exactly the coefficients for the series for $\frac{M}{1-z}$. By comparison, we see the Taylor series has radius of convergence at least $r$.

But how do we know that the Taylor series describes the function. We have one last trick at our disposal. We return to the formula:

$$f(w) = \frac{1}{2\pi i} \int_{\alpha} \frac{f(z)dz}{z-w}.$$ 

Notice that

$$\frac{1}{z-w} = \frac{1}{z} \frac{1}{1 - \frac{z}{w}} = \frac{1}{z} \left(1 + \frac{z}{w} + \left(\frac{z}{w}\right)^2 + \ldots\right).$$

By expanding the integrand in a series in $w$, we get exactly our convergent Taylor series.

The conclusion is that all the elementary functions of calculus and indeed all analytic functions have convergent power series. Indeed all the fancy complex calculus we’ve been doing with them could have just been concluded from that. (In the nineteenth century, people tried to expand the list of elementary functions by adding in their favorite “special functions” which they defined as power series. But much of this knowledge is now largely forgotten.)

We had to pass to complex numbers to make these arguments. When we dealt with Taylor series for real function, we could finite Taylor approximations, but under very reasonable conditions like $C^\infty$, we had no guarantee of a Taylor series, even if it converged, converged to the value of the function. The complex numbers overshadow the elementary calculus of one variable silently pulling its strings.

With that revelation, we close the book on Math 1a.