In the previous lecture, we used the least upper bound property of the real numbers to define the basic arithmetic operations of addition and multiplication. In effect, this involved finding sequences which converged to the sum and product. In the current lecture, we will make the notion of convergence to a limit by a sequence more flexible and more precise.

In general, a sequence of real numbers is a set of real numbers \( \{a_n\} \) which is indexed by the natural numbers. That is each element of the sequence \( a_n \) is associated to a particular natural number \( n \). We will refer to \( a_n \) as the \( n \)th element of the sequence.

**Example 1** Let \( x \) and \( y \) be positive real numbers. Let \( t_n(x) \) and \( t_n(y) \) be as defined in Lecture 2, the truncations of \( x \) and \( y \) to their decimal expansions up to \( n \) places. Consider the sequence \( \{a_n\} \) given by
\[
a_n = t_n(x)t_n(y).
\]
The number \( a_n \) represents the approximation to the product \( xy \) obtained by neglecting all contributions coming from after the \( n \)th decimal place. The sequence \( a_n \) is increasing meaning that if \( n > m \) then \( a_n \geq a_m \).

**Example 2** Consider the sequence \( \{b_n\} \) given by
\[
b_n = 1 + (-2)^{-n}.
\]
The sequence \( \{b_n\} \) is neither increasing nor decreasing since the odd elements of the sequence are less than one while the even ones are greater than one.

As a proxy for taking a limit of the sequence in Example 1, when we studied it in lecture 2, we took the least upper bound. But this only worked because the sequence was increasing. In example 2, the least upper bound is \( \frac{5}{4} \) since we have written 0 out of the natural numbers. The greatest lower bound is \( \frac{1}{2} \). But the limit should be 1 since the sequence oscillates ever closer to 1 as \( n \) increases. In what follows, we will write down a definition of convergence to limits under which both sequences converge. (This should not be too surprising.) Indeed, since you have already had Calculus, you probably have a good sense about which sequences converge. Nevertheless, you should pay careful attention to the definition of convergence, because though it is technical, it contains within it a way of quantifying not just whether sequences converge but how fast. This is information of great practical importance.

**Definition** We say that the sequence \( \{a_n\} \) converges to the limit \( L \) if for every real number \( \epsilon > 0 \) there is a natural number \( N \) so that \( |a_n - L| < \epsilon \) whenever \( n > N \). We sometimes denote \( L \) by
\[
\lim_{n \to \infty} a_n.
\]

**Example 3**: The limit of the sequence in Example 2 Let \( \{b_n\} \) be as before. We will show that the sequence \( \{b_n\} \) converges to the limit 1. We observe that \( |b_n - 1| = 2^{-n} \).
To complete our proof, we must find for each real number $\epsilon > 0$, a natural number $N$ so that when $n > N$, we have that $2^{-n} < \epsilon$. Since the error $2^{-n}$ decreases as $n$ increases, it is enough to ensure that $2^{-N} < \epsilon$. We can do this by taking $N > \log_2 \frac{1}{\epsilon}$. A note for sticklers: if we want that to be completely rigorous, maybe we have to verify that $\log_2$ is defined for all real numbers which we haven’t done yet. However, it is quite easy to see that for any $\epsilon > 0$, there is an $m$ so that $10^{-m} < \epsilon$. This is readily done since $\epsilon$ being nonzero and positive has a nonzero digit in its decimal expansion and we can choose $m$ to be the place of that digit. Then we can use an inequality like $2^{-4m} = 16^{-m} < 10^{-m}$.

All you have who have studied proofs, for instance in the online course Math 0, are probably under the impression that proofs are a bunch of verbiage used to certify some trivial fact that we already know. They have to be written in grammatical complete sentences and follow basic rules of logic. All of this can be said to be true about proofs that limits exist. But there is one additional element that you have to supply in order to prove a limit exists. You have to find a function $N(\epsilon)$. (Since the number $N$ depends on the number $\epsilon$.) What does this function mean? Here $\epsilon$ is a small number. It is the error one is willing to tolerate between a term in the sequence and the limit of the sequence. Then $N(\epsilon)$ represents how far we need to go in the sequence until we are certain that the terms in the sequence will approximate the limit to our tolerance. This can be very much a practical question. For instance in Example 1, the terms in the sequence are approximations to the product $xy$ which can be calculated in a finite number of steps. Recall that 7th grade textbooks insist that real numbers be multiplied by calculators which seems ridiculous since calculators can only show numbers up to some accuracy $\epsilon$ (which used to be $10^{-8}$.) In order for the calculator to comply with the wishes of the 7th grade textbook it needs to know $N(10^{-8})$ so that it will know how far in the sequence it has to go to get an answer with appropriate accuracy.

Thus the function $N(\epsilon)$ is really important. It is strange that it is so easy for you to think that proofs that limits exist only answer questions you already know the answer to. This is because you all have superb intuition as to whether limits exist. But why does the question always have to be “does the limit exist?” Couldn’t it equally well be, “What is a function $N(\epsilon)$ which certifies that the limit exists?” Probably, it is because of deep anti-mathematical biases which exist in society. After all, the first question has a unique yes or no answer. For the second question, the answer is a function and it is not unique. In fact, given a function that works, any larger function also works. Of course, it can also be said that answers to the second question, even if correct, do not have equal value. It is better for the calculator to get as small a function as it can that it can guarantee works.

You will be required to prove limits exist in this course. Because you’ve had the opportunity to take Math 0 and are intelligent human beings, I won’t spend any time instructing you in how to write grammatical sentences or how to reason logically. But a really legitimate question for you to be asking yourselves is “How do I find a function $N(\epsilon)$?” The most obvious thing to say is that you should be able to estimate the error between the $N$th term of the sequence and the limit. If this error is decreasing, you have already found the inverse function for $N(\epsilon)$ and just have to invert. (This is what happened
in Example 3: the function $\log_2(\frac{1}{2})$ is the inverse of the function $2^{-n}$. If the errors aren’t always decreasing, you may have to get an upper bound on all later errors too.

The question still remains how do we find these upper bounds. Therein lies the artistry of the subject. Because we are estimating the difference between a limit and a nearby element of a sequence, there is often a whiff of differential calculus about the process. This may seem ironic since we have not yet established any of the theorems of differential calculus and this is one of our goals for the course. Nevertheless, your skills at finding derivatives, properly applied, may prove quite useful.

Example 4: the limit of the sequence in Example 1. We would like to show that for $x$ and $y$ positive real numbers, the sequence $\{t_n(x)t_n(y)\}$ converges to the product $xy$ which is defined as the least upper bound of the sequence. Thus we need to estimate $|xy - t_n(x)t_n(y)| = xy - t_n(x)t_n(y)$. We observe that $t_n(x)t_n(y) \leq xy \leq \left(t_n(x) + \frac{1}{10^n}\right)\left(t_n(y) + \frac{1}{10^n}\right)$, since $t_n(x) + \frac{1}{10^n}$ has a larger $n$th place than any truncation of $x$ and similarly for $y$. Now subtracting $t_n(x)t_n(y)$ from the inequality, we get

$$0 \leq xy - t_n(x)t_n(y) \leq \left(t_n(x) + \frac{1}{10^n}\right)\left(t_n(y) + \frac{1}{10^n}\right) - t_n(x)t_n(y).$$

Note that the right hand side looks a lot like the expressions one gets from the definition of the derivative, where $\frac{1}{10^n}$ plays the role of $h$. Not surprisingly then, when we simplify, what we get is reminiscent of the product rule

$$0 \leq xy - t_n(x)t_n(y) \leq \frac{1}{10^n}(t_n(x) + t_n(y) + \frac{1}{10^n} - t_n(x)t_n(y)).$$

Notice we are free to use the distributive law because we are only applying it to rational numbers. The last step is a little wasteful, but we have done it to have a function that is readily invertible. Clearly $\frac{1}{10^n}(x + y + 1)$ is decreasing as $n$ increases. Thus if we just solve for $N$ in

$$\epsilon = \frac{1}{10^n}(x + y + 1),$$

we find the function $N(\epsilon)$. It is easy to see that $N(\epsilon) = \log_{10} \frac{x+y+1}{\epsilon}$ works. To summarize the logic, when $n > N(\epsilon)$ then

$$|xy - t_n(x)t_n(y)| \leq \frac{1}{10^n}(x + y + 1) \leq \epsilon.$$

Thus we have shown that $t_n(x)t_n(y)$ converges to the limit $xy$.

A clever reader might think that the hard work of Example 4 is really unnecessary. Shouldn’t we know just from the fact that the sequence is increasing that it must converge to its least upper bound? This is in fact the case.
**Theorem 1** Let \( \{a_n\} \) be an increasing sequence of real numbers which is bounded above. Let \( L \) be the least upper bound of the sequence. Then the sequence converges to the limit \( L \).

**Proof** We will prove Theorem 1 by contradiction. We suppose that the sequence \( \{a_n\} \) does not converge to \( L \). This means there is some real number \( \epsilon > 0 \) for which there is no \( N \), so that when \( n > N \), we are guaranteed that \( L - \epsilon \leq a_n \leq L \). This means there are arbitrarily large \( n \) so that \( a_n < L - \epsilon \). But since \( a_n \) is an increasing sequence, this means that all \( a_n < L - \epsilon \), since we can always find a later term in the sequence, larger than \( a_n \) which is smaller than \( L - \epsilon \). We have reached a contradiction since this means that \( L - \epsilon \) is an upper bound for the sequence so \( L \) could not have been the least upper bound.

A direct application of Theorem 1 shows that the limit of example 1 converges. Is the clever reader right that Example 4 is unnecessary? Not necessarily. A practical reader should object that the proof through Theorem 1 is entirely unquantitative. It doesn’t give us an explicit expression for \( N(\epsilon) \) and so it doesn’t help the calculator one iota. It provides no guarantee of when the approximation is close to the limit. Mathematicians are known for looking for elegant proofs, where elegant is usually taken to mean short. In this sense, the proof through Theorem 1 is elegant. That doesn’t necessarily make it better. Sometimes if you’re concerned about more than what you’re proving, it might be worthwhile to have a longer proof, because it might give you more information.

**Example 5: The distributive law** Last time, we stopped short of proving the distributive. Let \( x, y, z \) be positive real numbers. We would like to show that \((x + y)z = xz + yz\). Precisely, this means that we want to show that

\[
\lim_{n \to \infty} t_n(x+y)t_n(z) = \lim_{n \to \infty} t_n(x)t_n(z) + \lim_{n \to \infty} t_n(y)t_n(z).
\]

If \( L_1 = (x + y)z \), \( L_2 = xz \), and \( L_3 = yz \), then these are the limits in the equality above. From the definition of the limit, we can find an \( N_1 \) so that for \( n > N \) the following three inequalities hold:

\[
|L_1 - t_n(x+y)t_n(z)| \leq \frac{\epsilon}{4}.
\]
\[
|L_2 - t_n(x)t_n(z)| \leq \frac{\epsilon}{4}.
\]
and
\[
|L_3 - t_n(y)t_n(z)| \leq \frac{\epsilon}{4}.
\]

Basically, we find an \( N \) for each inequality and take \( N_1 \) to be the largest of the three. To get \( N_1 \) explicitly, we can follow example 4.

Next we observe that

\[
(t_n(x) + t_n(y))t_n(z) \leq t_n(x+y)t_n(z) \leq (t_n(x) + t_n(y) + \frac{2}{10n})t_n(z),
\]

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since the right hand side is more than \( x + y \). Thus
\[
(t_n(x) + t_n(y))t_n(z) \leq t_n(x + y)t_n(z) \leq (t_n(x) + t_n(y) + \frac{2}{10^n})t_n(z),
\]
from which we can conclude (again following the ideas of Example 4) that there is \( N_2 \) so that when \( n > N_2 \) we have
\[
|t_n(x + y)t_n(z) - (t_n(x) + t_n(y)t_n(z))| \leq \frac{\epsilon}{4}.
\]
Now take \( N \) to be the maximum of \( N_1 \) and \( N_2 \). We have shown that when we go far enough in each sequence past \( N \), the terms in limiting sequences to \( L_1, L_2 \), and \( L_3 \) are within \( \frac{\epsilon}{4} \) of the limits and that the difference between the \( n \)th term in the sequence for \( L_1 \) and the sum of the \( n \)th terms for \( L_2 \) and \( L_3 \) is at most \( \frac{\epsilon}{4} \). Combining all the errors, we conclude that
\[
|L_1 - L_2 - L_3| \leq \epsilon.
\]
Since \( \epsilon \) is an arbitrary positive real number and absolute values are nonnegative, we conclude that \( L_1 - L_2 - L_3 = 0 \), which is what we were to show.

It is worth noting that when combined the errors, we were in effect applying the triangle inequality
\[
|a - c| \leq |a - b| + |b - c|
\]
multiple times. This inequality holds for all reals \( a, b, c \). See the homework.

At this point, we are in a position to establish for the reals all arithmetic identities that we have for the rationals in the spirit of Example 5. Basically we approximate any quantity we care about closely enough by terminating decimal expansions and we can apply the identity for the rationals. For this reason, we will not have much further need to refer to decimal expansions in the course. We have established what the real numbers are and that they do what we expect of them. Moreover, we have seen how to use the formal definition of the limit and what it means. Next time, we will discuss additional criteria under which we are guaranteed that a limit exists.