A very important class of series to study are the power series. They are interesting in part because they represent functions and in part because they encode their coefficients which are a sequence. At the end of this lecture, we will see an application of power series for writing a formula for an interesting sequence.

**Definition** A power series is an expression of the form

\[ S(x) = \sum_{j=0}^{\infty} a_j x^j. \]

For the moment, the coefficients \( a_j \) will be real numbers. The variable \( x \) takes real values and for each distinct value of \( x \), we get a different series \( S(x) \). The first question we’ll be interested in is for what values of \( x \) does the series \( S(x) \) converge.

**Theorem 1** Let

\[ S(x) = \sum_{j=0}^{\infty} a_j x^j. \]

Then there is a unique \( R \in [0, \infty] \) so that \( S(x) \) converges absolutely when \( |x| < R \) and so that \( S(x) \) diverges when \( |x| > R \).

The number \( R \) (possibly infinite) which Theorem 1 guarantees is called the radius of convergence of the power series.

Often to prove a theorem, we break it down into simpler parts which we call Lemmas. This is going to be one of those times.

**Lemma 1** Let

\[ S(x) = \sum_{j=0}^{\infty} a_j x^j. \]

Suppose that \( S(c) \) converges. Then \( S(x) \) converges absolutely for all \( x \) so that \( |x| < |c| \).

**Proof of Lemma 1** We note that since \( S(c) \) converges, it must be that the sequence of numbers \( \{|a_j c^j|\} \) are bounded above. If not, there are arbitrarily large partial sums of \( S(c) \) which differ by an arbitrarily large quantity, so that the series \( S(c) \) does not converge. Let \( K \) be an upper bound for the sequence \( \{|a_j c^j|\} \). Now suppose \( |x| < |c| \). We will show \( S(x) \) converges absolutely. Observe that we have the inequality

\[ |a_j x^j| \leq K \left( \frac{x}{c} \right)^j. \]
Thus by Theorem 2 of Lecture 5, it suffice to show that the series
\[ \sum_{j=0}^{\infty} K |(\frac{x}{c})|^j \]
converges. But this is true since the series above is geometric and by assumption $|\frac{x}{c}| < 1$.

Now we are in a strong position to prove Theorem 1.

**Proof of Theorem 1** We will prove theorem 1 by defining $R$. We let $R$ be the least upper bound of the set of $|x|$ so that $S(x)$ converge. If this set happens not to be bounded above, we let $R = \infty$. By the definition of $R$, it must be that for any $x$ with $|x| > R$, we have that $S(x)$ diverges. (Otherwise $R$ isn’t an upper bound.) Now suppose that $|x| < R$. Then there is $y$ with $|y| > |x|$ so that $S(y)$ converges. (Otherwise, $|x|$ is an upper bound.) Now, we just apply Lemma 1 to conclude that $S(x)$ converges.

The above proof gives the radius of convergence $R$ in terms of the set of $x$ where the series converges. We can however determine it in terms of the coefficients of the series. We consider the sets
\[ A_k = \{|a_n|^\frac{1}{n} : n \geq k\} \]
These are the sets of $n$th roots of $n$th coefficients in the tail of the series. Let $T_k$ be the least upper bound of $A_k$. The numbers $T_k$ are a decreasing sequence of positive numbers and have a limit unless they are all infinite. Let
\[ T = \lim_{k \to \infty} T_k. \]
Then $T$ is a nonnegative real number or is infinite. It turns out that $R = \frac{1}{T}$. You will be asked to show this on the homework, but it is a rather simple application of the $n$th root test. This is the reason the $n$th root test is important for understanding power series.

One thing we haven’t discussed yet is the convergence of the power series right at the radius of convergence. Basically, all outcomes are possible. Directly at the radius of convergence, we are in a setting where the $n$th root test fails.

**Example 1** Consider the following three series.
\[ S_1(x) = \sum_{n=0}^{\infty} x^n. \]
\[ S_2(x) = \sum_{n=1}^{\infty} \frac{x^n}{n}. \]
\[ S_3(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}. \]
By the criterion above, it is rather easy to see that the radius of convergence of each series is 1, since the $n$th roots of the coefficients converge to 1. However the three series have rather different behaviors at the points $x = 1$ and $x = -1$. We note that $S_1(x)$ diverges at both $x = 1$ and $x = -1$ since all of its terms there have absolute value 1. We note that $S_2(1)$ is the harmonic series which diverges and we note that $S_2(-1)$ is the alternating version of the harmonic series which we showed converges conditionally. We can see that $S_3(1)$ and $S_3(-1)$ both converge absolutely since the can be compared with the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$ 

Since we are interested in studying power series as functions and we are accustomed to adding and multiplying functions, it will be important to us to understand that we can add and multiply absolutely convergent series termwise. Once we have done this, we will see that we can do the same with power series inside their radius of convergence.

**Theorem 2** Let $S_1 = \sum_{n=0}^{\infty} a_n$ and $S_2 = \sum_{n=0}^{\infty} b_n$ be absolutely convergent series. Then

$$S_1 + S_2 = \sum_{n=0}^{\infty} a_n + b_n,$$

and letting

$$c_m = \sum_{i+j=m} a_i b_j,$$

we have

$$S_1 S_2 = \sum_{n=0}^{\infty} c_n.$$ 

It is worth noting that even the statement of the theorem for products looks a little more complicated than for sums. The issue is that (in the case of power series) products of partial sums are not exactly the partial sums of the products.

**Proof of Theorem 2** The proof of the statement about sums is essentially immediate since the partial sums of the formula for sum are the sums of partial sums of the individual series. So we need only check that the limit of a sum is the sum of the limits, which we leave to the reader. For products, things are a little more complicated. We observe that the sum of an absolutely convergent series is the difference between the sum of the series of its positive terms and the sum of the series of its negative terms and so we restrict our attention to the case where all $a_i$’s and all $b_i$’s are nonnegative. We let $S_{1,n}$ be the $n$th partial sum of $S_1$ and $S_{2,n}$ be the $n$th partial sum of $S_2$ and we let $S_{3,n}$ be the $n$th partial sum of

$$\sum_{n=0}^{\infty} c_n.$$ 

Then we notice that
\[ S_{1,n}S_{2,n} \leq S_{3,2n} \leq S_{1,2n}S_{2,2n}. \]
We obtain the desired conclusion using the Squeeze theorem.

In a few weeks, when we study Taylor’s theorem, we will establish power series expressions for essentially all the functions that we know how to differentiate. As it is, we already know power series expansions for a large class of functions because of our familiarity with geometric series.

**Example 2**

\[ \frac{1}{1 - ax} = \sum_{n=0}^{\infty} (ax)^n, \]
whenever \(|ax| < 1\). The equality expressed above is just a special case of the formula for the sum of an infinite geometric series. However the right hand side is a power series expression for the function on the left hand side. The radius of convergence of the series is \( \frac{1}{|a|} \) which is the distance from zero to the singularity of the function.

In conjunction with Theorem 2, we can actually use this formula to obtain the power series at zero of any rational function. Suppose

\[ f(x) = \frac{P(x)}{Q(x)} \]

is a rational function (that is \( P(x) \) and \( Q(x) \) are polynomials.) Suppose moreover that the roots of \( Q(x) \) are distinct. Let us call them \( r_1, \ldots, r_m \), then by partial fractions decomposition

\[ f(x) = S(x) + \frac{A_1}{x - r_1} + \ldots + \frac{A_m}{x - r_m}, \]

where \( S(x) \) is a polynomial and the \( A \)'s are constants. Using geometric series, we already have a series expansion for each term in this sum.

What happens if \( Q(x) \) does not have distinct roots. Then we need power series expansions for \( \frac{1}{(x - r_1)^2}, \frac{1}{(x - r_2)^2}, \ldots \). In a few weeks, we’ll see that an easy way of getting them is by differentiating the series for \( \frac{1}{x - r} \). But as it is, we can also get the series by taking \( \frac{1}{x - r} \) to powers. For instance,

\[ \frac{1}{(1 - ax)^2} = \left( \sum_{n=0}^{\infty} (ax)^n \right)^2 = \sum_{n=0}^{\infty} (n+1)(ax)^n. \]

Here what we have done is simply apply the multiplication part of Theorem 2. As long as we can count the number of terms in the product, we are now in a position to obtain a series expansion for any rational function.
**Example 3** As promised, we will now use the theory of power series to understand the terms of an individual sequence. We now define the Fibonacci sequence. It is defined by letting \( f_0 = 1 \) and \( f_1 = 1 \). Then for \( j \geq 2 \), we let

\[
f_j = f_{j-1} + f_{j-2}.
\]

The above formula is called the recurrence relation for the Fibonacci sequence and it lets us generate this sequence one term at a time:

\[
f_0 = 1, \quad f_1 = 1, \quad f_2 = 2, \quad f_3 = 3, \quad f_4 = 5, \quad f_5 = 8, \quad f_6 = 13, \quad f_7 = 21, \ldots
\]

The Fibonacci sequence is much loved by math geeks and has a long history. It was first used by Fibonacci in the eighth century to model populations of rabbits for reasons that are too upsetting to relate.

Nevertheless our present description of the sequence is disturbingly inexplicit. To get each term, we need first to have computed the previous two terms. This situation is sufficiently alarming that the world’s bestselling Calculus book gives the Fibonacci sequence as an example of a sequence whose \( n \)th term cannot be described by a simple formula. Using power series, we are now in a position to make a liar of that Calculus book.

We introduce the following power series

\[
f(x) = \sum_{n=0}^{\infty} f_n x^n,
\]

which has the Fibonacci sequence as its coefficients. We note that multiplying \( f(x) \) by a power of \( x \) shifts the sequence. We consider the expression \((1 - x - x^2)f(x)\) and note that by the recurrence relation, all terms with \( x^2 \) or higher vanish. Computing the first two terms by hand, we see that

\[
(1 - x - x^2)f(x) = 1,
\]

or put differently

\[
f(x) = \frac{1}{1 - x - x^2}.
\]

Apply partial fractions, we conclude

\[
f(x) = -\frac{1}{\sqrt{5}} \frac{1}{x + \frac{1 + \sqrt{5}}{2}} + \frac{1}{\sqrt{5}} \frac{1}{x + \frac{1 - \sqrt{5}}{2}}.
\]

Now applying the formula for sum of a geometric series and using the fact that

\[
\left(\frac{1 + \sqrt{5}}{2}\right)\left(\frac{1 - \sqrt{5}}{2}\right) = -1,
\]

we see that

\[
f_n = \frac{1}{\sqrt{5}} \left(\frac{1 + \sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}} \left(\frac{1 - \sqrt{5}}{2}\right)^n.
\]

What could be simpler than that?